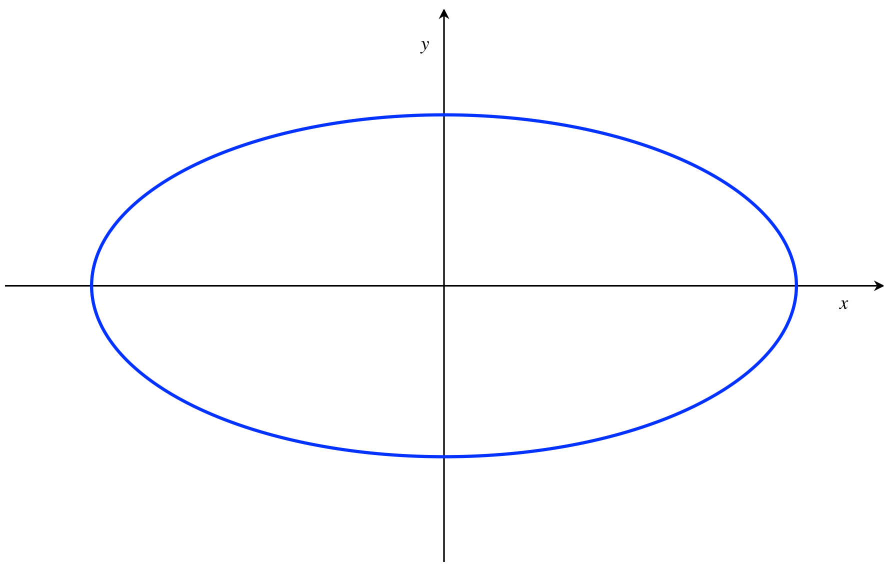
Math 2260 Written HW #8 Solutions

1. Suppose a and b are two fixed positive numbers. Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

[*Hint: you might want to find the area of the half of the ellipse above the x-axis and then double it.*]

Answer: Here is a picture of the ellipse, which has x-intercepts at $x = \pm a$ and y-intercepts at $y = \pm b$.



I can find the area of the top half of the ellipse and then double it. To do so, I'll solve for y as a function of x. Certainly,

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2},$$

so

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

and hence

$$y = \pm\sqrt{b^2\left(1 - \frac{x^2}{a^2}\right)} = \pm b\sqrt{1 - \frac{x^2}{a^2}}.$$

The top half of the ellipse corresponds to choosing the positive square root in the above. For later convenience, notice that $1 - \frac{x^2}{a^2} = \frac{1}{a^2}\left(a^2 - x^2\right)$, and hence the top half of the ellipse is given by

$$y = \frac{b}{a}\sqrt{a^2 - x^2}.$$

Since the ellipse extends from $x = -a$ to $x = a$, the area under the above curve (which is the area of the top half of the ellipse) is equal to

$$\int_{-a}^{a} \frac{b}{a}\sqrt{a^2 - x^2}\,dx.$$

Therefore, the area of the whole ellipse is equal to

$$2\int_{-a}^{a} \frac{b}{a}\sqrt{a^2 - x^2}\,dx.$$

To evaluate this integral, make the trig substitution $x = a\sin\theta$. Then $dx = a\cos\theta\,d\theta$. Also, since $\sin\theta = x/a$, we see that $\sin\theta = -a/a = -1$ when $x = -a$, meaning that $\theta = -\pi/2$. Likewise, when $x = a$, $\sin\theta = a/a = 1$, so $\theta = \pi/2$. Putting this all together, then, the above integral is equal to

$$\begin{aligned}
2\frac{b}{a}\int_{-\pi/2}^{\pi/2} \sqrt{a^2 - a^2\sin^2\theta}\cdot a\cos\theta\,d\theta &= 2\frac{b}{a}\int_{-\pi/2}^{\pi/2} \sqrt{a^2\cos^2\theta}\cdot a\cos\theta\,d\theta \\
&= 2\frac{b}{a}\int_{-\pi/2}^{\pi/2} a^2\cos^2\theta\,d\theta \\
&= 2ab\int_{-\pi/2}^{\pi/2} \frac{1+\cos(2\theta)}{2}\,d\theta \\
&= ab\int_{-\pi/2}^{\pi/2} \left(1+\cos(2\theta)\right)\,d\theta \\
&= ab\left[\theta + \frac{\sin(2\theta)}{2}\right]_{-\pi/2}^{\pi/2} \\
&= ab\left[\left(\frac{\pi}{2}+0\right) - \left(\frac{\pi}{2}+0\right)\right] \\
&= \pi ab.
\end{aligned}$$

Therefore the area of the ellipse is πab.

2. Evaluate the integral

$$\int \frac{e^{4t} + 2e^{2t} - e^t}{e^{2t}+1}\,dt.$$

Answer: First, make the u-substitution $u = e^t$. Then $du = e^t\,dt$ and so the above integral is equal to

$$\int \frac{\left(e^{3t} + 2e^t - 1\right)e^t\,dt}{e^{2t}+1} = \int \frac{u^3 + 2u - 1}{u^2+1}\,du.$$

Dividing numerator by denominator yields

$$\begin{array}{r|l}
 & u \\
u^2 + 0u + 1 & u^3 + 0u^2 + 2u - 1 \\
 & \underline{u^3 + 0u^2 + u \phantom{{}-1}} \\
 & \phantom{u^3 + 0u^2 + {}} u - 1
\end{array}$$

so our integral is equal to

$$\int \left(u + \frac{u-1}{u^2+1}\right) du = \int \left(u + \frac{u}{u^2+1} - \frac{1}{u^2+1}\right) du = \int u\, du + \int \frac{u}{u^2+1}\, du - \int \frac{1}{u^2+1}\, du.$$

The first integral is easy. For the second, let $v = u^2 + 1$, then $dv = 2u\, du$ and so

$$\int \frac{u}{u^2+1}\, du = \frac{1}{2}\int \frac{dv}{v} = \frac{1}{2}\ln v + C_1 = \frac{1}{2}\ln(u^2+1) + C_1$$

.

For the third integral, let $u = \tan\theta$. Then $du = \sec^2\theta\, d\theta$ and so

$$\int \frac{1}{u^2+1}\, du = \int \frac{1}{\tan^2\theta + 1}\sec^2\theta\, d\theta = \int \frac{\sec^2\theta}{\sec^2\theta}\, d\theta = \theta + C_2 = \arctan(u) + C_2.$$

Therefore, the above u-integral is equal to

$$\frac{u^2}{2} + \frac{1}{2}\ln(u^2+1) - \arctan u + C,$$

where I've combined all the constants into the single constant C. Now, since $u = e^t$, this means that the original integral is equal to

$$\frac{e^{2t}}{2} + \frac{1}{2}\ln(e^{2t}+1) - \arctan(e^t) + C.$$

3. Sociologists sometimes use the phrase "social diffusion" to describe the way information spreads through a population. The information could be anything: a scientific breakthrough, news of a natural disaster, literacy, etc. In a sufficiently large population, the number of people x who have the information is treated as a differentiable function of time t, and the rate of diffusion, dx/dt, is assumed to be proportional to the number of people who have the information times the number of people who do not. This leads to the equation

$$\frac{dx}{dt} = kx(N-x),$$

where N is the number of people in the population. The constant k determines how fast the information spreads: you would expect k to be relatively large for, say, celebrity gossip, which is assimilated almost instantly, whereas it would be quite small for information that only disseminates very slowly (like how to compute integrals!).

Suppose t is in days, $k = 1/250$, and two people start a rumor at time $t = 0$ in a population of $N = 1000$ people.

(a) Find x as a function of t. [*Hint: Logarithm identities are your friends!*]

Answer: First, we separate variables and integrate:

$$\int \frac{dx}{x(1000-x)} = \int \frac{1}{250}\, dt. \qquad (*)$$

The right hand side will be easy, so let's focus on the left hand side. Using the method of partial fractions, we guess that

$$\frac{1}{x(1000-x)} = \frac{A}{x} + \frac{B}{1000-x}.$$

Clearing denominators yields

$$1 = A(1000-x) + Bx,$$

or, equivalently,

$$1 = (B-A)x + 1000A,$$

which yields the system of equations

$$\begin{aligned} 0 &= B - A \\ 1 &= 1000A. \end{aligned}$$

The second equation implies that $A = \frac{1}{1000}$, while the first implies that $B = A = \frac{1}{1000}$. Therefore,

$$\int \frac{dx}{x(1000-x)} = \int \left(\frac{1/1000}{x} + \frac{1/1000}{1000-x} \right) dx = \frac{1}{1000} \ln x - \frac{1}{1000} \ln(1000-x)$$

(we don't have to worry about absolute values, because we're only interested in values of x between 0 and 1000).

Substituting this into equation ($*$) and integrating the right hand side gives us

$$\frac{1}{1000} \ln x - \frac{1}{1000} \ln(1000-x) = \frac{1}{250} t + C.$$

Multiply both sides by 1000 to get

$$\ln x - \ln(1000-x) = 4t + 4C.$$

Now here's where the logarithm identity $\ln a - \ln b = \ln \left(\frac{a}{b} \right)$ comes in handy. We can use this identity to re-write the left hand side:

$$\ln \left(\frac{x}{1000-x} \right) = 4t + 4C.$$

Since we want to solve for x, exponentiate both sides to get

$$\frac{x}{1000-x} = e^{4t+4C} = e^{4C} e^{4t} = Ae^{4t},$$

where I've let $A = e^{4C}$. This is still a little unpleasant, but it gets nicer if I take the reciprocal of both sides:

$$\frac{1000-x}{x} = \frac{1}{Ae^{4t}}.$$

The left hand side is equal to $\frac{1000}{x} - 1$, so I have that

$$\frac{1000}{x} = \frac{1}{Ae^{4t}} + 1 = \frac{1 + Ae^{4t}}{Ae^{4t}}$$

(where I found a common denominator and added the terms on the right hand side). Taking the reciprocal again yields

$$\frac{x}{1000} = \frac{Ae^{4t}}{1 + Ae^{4t}},$$

and so

$$x = 1000\frac{Ae^{4t}}{1 + Ae^{4t}}.$$

Since the rumor started with two people, we know that $x(0) = 2$. We can use this to solve for the constant A:

$$x(0) = 1000\frac{Ae^{4(0)}}{1 + Ae^{4(0)}}$$
$$2 = 1000\frac{A}{1 + A}$$

Dividing both sides by 1000, we have

$$\frac{1}{500} = \frac{A}{1 + A}.$$

To solve for A, take the reciprocal of both sides:

$$500 = \frac{1 + A}{A} = \frac{1}{A} + 1,$$

so

$$\frac{1}{A} = 499$$

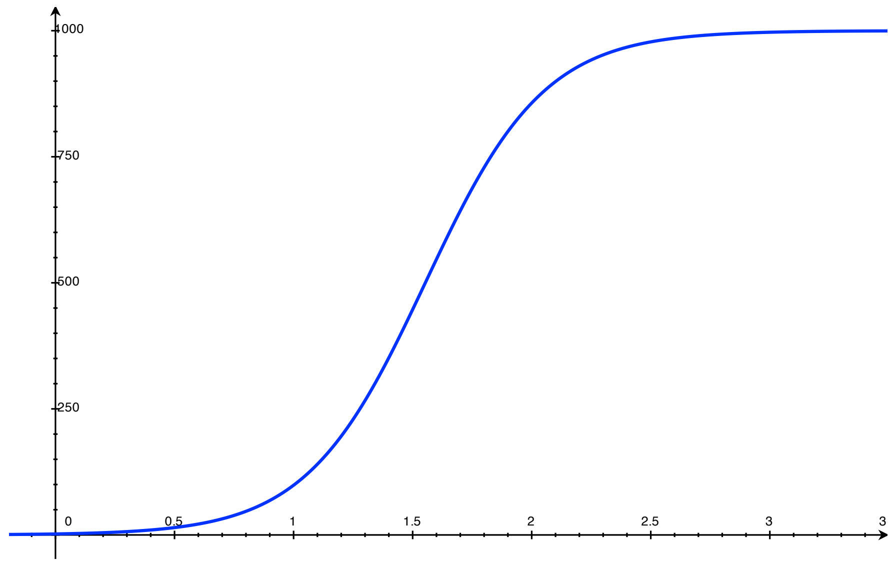
and hence

$$A = \frac{1}{499}.$$

Finally, then, this means that

$$x(t) = 1000\frac{\frac{1}{499}e^{4t}}{1 + \frac{1}{499}e^{4t}} = 1000\frac{e^{4t}}{499 + e^{4t}}.$$

Here's a graph x as a function of t:



(b) When will half the population have heard the rumor? (This is when the rumor will be spreading the fastest.)
Answer: The question is: at what time t_0 will we have that $x(t_0) = 500$? Well, evaluate the function at $t = t_0$:

$$x(t_0) = 1000\frac{e^{4t_0}}{499 + e^{4t_0}}$$
$$500 = 1000\frac{e^{4t_0}}{499 + e^{4t_0}}.$$

Diving both sides by 1000 yields

$$\frac{1}{2} = \frac{e^{4t_0}}{499 + e^{4t_0}}.$$

Take the reciprocal of both sides to get

$$2 = \frac{499 + e^{4t_0}}{e^{4t_0}} = \frac{499}{e^{4t_0}} + 1,$$

so we have

$$1 = \frac{499}{e^{4t_0}}$$

and so

$$e^{4t_0} = 499.$$

Take the natural logarithm of both sides:

$$4t_0 = \ln(499),$$

and hence

$$t_0 = \frac{\ln 499}{4} \approx 1.55.$$

Therefore, it take about one and a half days for half the population to have heard the rumor.