Math 2260 Written HW #7 Solutions

1. (a) Use integration by parts to show that

$$\int (\ln x)^n \, dx = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx.$$

Answer: For the integration by parts, let $u = (\ln x)^n$ and let $dv = dx$. Then we have

$$\begin{aligned} u &= (\ln x)^n & dv &= dx \\ du &= n(\ln x)^{n-1}\frac{1}{x} & v &= x. \end{aligned}$$

Integrating by parts yields

$$\begin{aligned} \int (\ln x)^n \, dx &= x(\ln x)^n - \int x \left(n(\ln x)^{n-1}\frac{1}{x} \right) \, dx \\ &= x(\ln x)^n - n \int (\ln x)^{n-1} \, dx, \end{aligned}$$

as desired.

(b) Use part (a) to evaluate the integral

$$\int (\ln x)^2 \, dx.$$

Answer: Using part (a) with $n = 2$, we know that

$$\int (\ln x)^2 \, dx = x(\ln x)^2 - 2 \int \ln x \, dx. \qquad (*)$$

Now, if we we part (a) on the second term with $n = 1$, we have

$$\begin{aligned} \int \ln x \, dx &= x \ln x - \int (\ln x)^0 \, dx \\ &= x \ln x - \int dx \\ &= x \ln x - x + C. \end{aligned}$$

Substituting this into (*) gives us

$$\begin{aligned} \int (\ln x)^2 \, dx &= x(\ln x)^2 - 2\left[x \ln x - x\right] + C \\ &= x(\ln x)^2 - 2x \ln x + 2x + C. \end{aligned}$$

2. Find the area of the region between the x-axis and the curve $y = \sqrt{1 + \cos(4x)}$ for $0 \le x \le \pi$. (*Careful! Remember that* $\sqrt{f(x)^2} = |f(x)|$.)

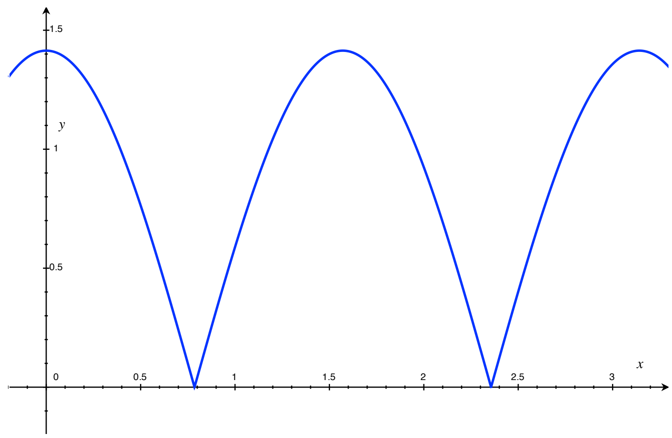
Answer: Notice that, by using the power-reduction formula backwards,

$$1 + \cos(4x) = 2\cos^2(2x).$$

Therefore,

$$\int_0^\pi \sqrt{1+\cos(4x)}\,dx = \int_0^\pi \sqrt{2\cos^2(2x)}\,dx = \int_0^\pi \sqrt{2}|\cos(2x)|\,dx.$$

Here's a graph of the function $y = \sqrt{2}|\cos(2x)|$:



To deal with the absolute value, I could either split the integral up into three pieces (namely $[0, \pi/4]$, $[\pi/4, 3\pi/4]$, and $[3\pi/4, \pi]$), or just notice that the total area under the curve from 0 to π is four times the area under the curve from 0 to $\pi/4$. Since $\cos(2x)$ is non-negative on $[0, \pi/4]$, then,

$$\begin{aligned}
\int_0^\pi \sqrt{2}|\cos(2x)|\,dx &= 4\int_0^{\pi/4} \sqrt{2}|\cos(2x)|\,dx \\
&= 4\int_0^{\pi/4} \sqrt{2}\cos(2x)\,dx \\
&= 4\left[\sqrt{2}\frac{\sin(2x)}{2}\right]_0^{\pi/4} \\
&= 4\left[\sqrt{2}\frac{1}{2} - 0\right] \\
&= 2\sqrt{2},
\end{aligned}$$

so the area of the region between the x-axis and the curve $y = \sqrt{1+\cos(4x)}$ for $0 \le x \le \pi$ is $2\sqrt{2}$.

3. Evaluate the indefinite integral

$$\int \frac{dx}{x^2\sqrt{x^2+1}}.$$

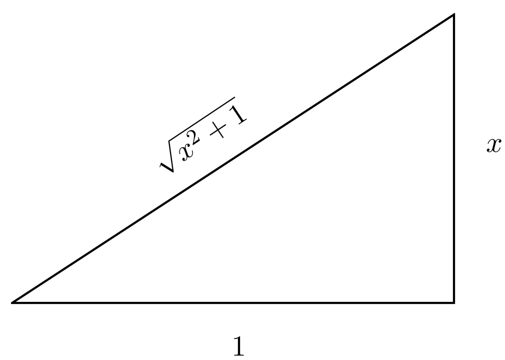
Answer: Since I see a term of the form $x^2 + a^2$, my first thought is to try a trigonometric substitution. In this case, since $a = 1$, I should let

$$\begin{aligned}
x &= \tan\theta \\
dx &= \sec^2\theta\,d\theta
\end{aligned}$$

Substituting that into the above integral, then, I have

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{x^2+1}} &= \int \frac{\sec^2\theta\,d\theta}{\tan^2\theta\sqrt{\tan^2\theta+1}}\\ &= \int \frac{\sec^2\theta\,d\theta}{\tan^2\theta\sqrt{\sec^2\theta}}\\ &= \int \frac{\sec\theta\,d\theta}{\tan^2\theta}\\ &= \int \frac{1}{\cos\theta}\frac{\cos^2\theta}{\sin^2\theta}\,d\theta\\ &= \int \frac{\cos\theta}{\sin^2\theta}\,d\theta\\ &= \int \frac{1}{\sin\theta}\frac{\cos\theta}{\sin\theta}\,d\theta\\ &= \int \csc\theta\cot\theta\,d\theta\\ &= -\csc\theta + C.\end{aligned}$$

Now, to convert back in terms of x, notice that $\tan\theta = x$, which means I can visualize θ as the bottom-left angle in the following right triangle:



Therefore, $\csc\theta = \frac{1}{\sin\theta} = \frac{\sqrt{x^2+1}}{x}$, and so I can conclude that

$$\int \frac{dx}{x^2\sqrt{x^2+1}}\,dx = -\frac{\sqrt{x^2+1}}{x} + C.$$