Math 2260 Written HW #6 Solutions

1. A lab receives a sample of polonium-210, which has a half-life of 139 days. The sample will be effectively useless by the time 95% percent of the radioactive nuclei present when the sample arrived have decayed. For how long will the lab be able to use the polonium?

   Answer: We know that, if $P(t)$ is the amount of polonium-210 in the sample after t days, then

   $$P(t) = P_0 e^{kt}.$$

   We're not given the initial amount of polonium (which would be P_0), but it doesn't matter since we're only interested in relative amounts. In particular, we're trying to figure out the time (which I'll call t_0) when the remaining polonium is 5% of the initial amount, which is to say when

   $$P(t_0) = 0.05 P_0 = \frac{P_0}{20}.$$

   First, though, I'll need to figure out the constant k. To do so, remember that the half-life is the amount of time it takes for half of the polonium to decay; in other words, $P(139) = \frac{P_0}{2}$. Therefore,

   $$\begin{aligned} P(139) &= P_0 e^{k \cdot 139} \\ \frac{P_0}{2} &= P_0 e^{139k}. \end{aligned}$$

   Dividing both sides by P_0 yields

   $$\frac{1}{2} = e^{139k}.$$

   Therefore, to solve for k I take the natural logarithm of both sides

   $$\ln\left(\frac{1}{2}\right) = 139k,$$

   remember that $\ln\left(\frac{1}{2}\right) = \ln 1 - \ln 2 = -\ln 2$, and then divide both sides by 139 to get that

   $$k = \frac{-\ln 2}{139}.$$

   Therefore, I have that

   $$P(t) = P_0 e^{\frac{-\ln 2}{139} t}.$$

   Now, remember that the goal was to find t_0 so that $P(t_0) = \frac{P_0}{20}$. Evaluating the above equation at t_0 yields

   $$\begin{aligned} P(t_0) &= P_0 e^{\frac{-\ln 2}{139} t_0} \\ \frac{P_0}{20} &= P_0 e^{\frac{-\ln 2}{139} t_0}. \end{aligned}$$

   Divide both sides by P_0 to get

   $$\frac{1}{20} = e^{\frac{-\ln 2}{139} t_0}$$

and then take the natural logarithm of both sides:

$$\ln\left(\frac{1}{20}\right) = \frac{-\ln 2}{139}\, t_0.$$

Since $\ln\left(\frac{1}{20}\right) = \ln 1 - \ln 20 = -\ln 20$, the above is the same as

$$-\ln 20 = \frac{-\ln 2}{139}\, t_0.$$

Hence, I can just multiply both sides by $\frac{139}{-\ln 2}$ to see that

$$t_0 = \frac{139 \ln 20}{\ln 2} \approx 600.7.$$

Therefore, the sample will cease to be useful after about 600 days.

2. An aluminum beam was brought from the outside cold into a machine shop where the temperature was held at 65°F. After 10 minutes, the beam warmed to 35°F, and after another 10 minutes it was 50°F. What was the beam's initial temperature?

   Answer: If $H(t)$ is the temperature of the beam after t minutes, then we know, by Newton's Law of Cooling, that

   $$H(t) = H_s + H_0 e^{kt},$$

   where H_s is the ambient temperature and H_0 and k are constants yet to be determined. In this case the ambient temperature is 65°F, so

   $$H(t) = 65 + H_0 e^{kt}.$$

   Ultimately, the goal is to determine the initial temperature of the beam, which is to say $H(0)$. To do so, we will need to determine the constants H_0 and k.

   Now, we have two additional facts at our disposal, namely that $H(10) = 35$ and that $H(20) = 50$. Using the first, I know that

   $$\begin{aligned} H(20) &= 65 + H_0 e^{k\cdot 10} \\ 35 &= 65 + H_0 e^{10k} \end{aligned}$$

   Therefore, $-30 = H_0 e^{10k}$. I can solve for k in terms of H_0 as follows. First, divide both sides by H_0 to get

   $$\frac{-30}{H_0} = e^{10k}.$$

   Then take the natural logarithm of both sides:

   $$\ln\left(\frac{-30}{H_0}\right) = 10k.$$

   Finally, divide both sides by 10 to get

   $$k = \frac{\ln\left(\frac{-30}{H_0}\right)}{10}.$$

Hence, I can re-write $H(t)$ as

$$H(t) = 65 + H_0 e^{\frac{\ln\left(\frac{-30}{H_0}\right)}{10} t}.$$

Now, evaluate at $t = 20$:

$$H(20) = 65 + H_0 e^{\frac{\ln\left(\frac{-30}{H_0}\right)}{10} \cdot 20}$$
$$50 = 65 + H_0 e^{2 \ln\left(\frac{-30}{H_0}\right)}$$

Subtracting 65 from both sides and using the fact that $a \ln(b) = \ln(b^a)$ yields

$$-15 = H_0 e^{\ln\left(\left(\frac{-30}{H_0}\right)^2\right)}$$
$$-15 = H_0 \left(\frac{-30}{H_0}\right)^2$$
$$-15 = H_0 \frac{900}{H_0^2}$$
$$-15 = \frac{900}{H_0}.$$

Now I just multiply both sides by $\frac{H_0}{-15}$ to see that

$$H_0 = \frac{900}{-15} = -60.$$

In turn, this means that $k = \frac{\ln\left(\frac{-30}{-60}\right)}{10} = \frac{\ln\left(\frac{1}{2}\right)}{10} = \frac{-\ln 2}{10}$, and so

$$H(t) = 65 - 60 e^{\frac{-\ln 2}{10} t}.$$

In particular, since we want to know $H(0)$, we can now just evaluate:

$$H(0) = 65 - 60 e^{\frac{-\ln 2}{10} \cdot 0} = 65 - 60 = 5,$$

so the beam started out at 5°F (hopefully nobody tried to touch it with their bare hands!).

3. Evaluate the indefinite integral

$$\int x^2 e^{-x}\, dx.$$

Answer: I intend to use integration by parts. Notice that neither integrating nor differentiating e^{-x} makes it any better or worse, so I can slot e^{-x} into either u or dv without any real worries. Since the goal is to choose u and dv so that the integral ultimately gets nicer, and since differentiating x^2 makes it better, whereas integrating x^2 makes it worse, I should choose $u = x^2$, which means $dv = e^{-x}\, dx$, so I have

$$u = x^2 \qquad dv = e^{-x}\, dx$$
$$du = 2x\, dx \qquad v = -e^{-x}$$

Therefore, using integration by parts, the given integral is equal to

$$-x^2e^{-x} - \int -e^{-x} \cdot 2x\,dx = -x^2e^{-x} + 2\int xe^{-x}\,dx. \tag{$*$}$$

To evaluate the integral in the second term, I will do integration by parts again. Following the pattern above, let $u = x$ and $dv = e^{-x}\,dx$, so I have

$$\begin{aligned} u &= x \qquad & dv &= e^{-x}\,dx \\ du &= dx \qquad & v &= -e^{-x} \end{aligned}$$

Therefore,

$$\begin{aligned} \int xe^{-x}\,dx &= -xe^{-x} - \int -e^{-x}\,dx \\ &= -xe^{-x} + \int e^{-x}\,dx \\ &= -xe^{-x} - e^{-x} + C_1 \end{aligned}$$

Combining this with ($*$), then, I see that the given integral is equal to

$$-x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C,$$

where I let $C = 2C_1$.