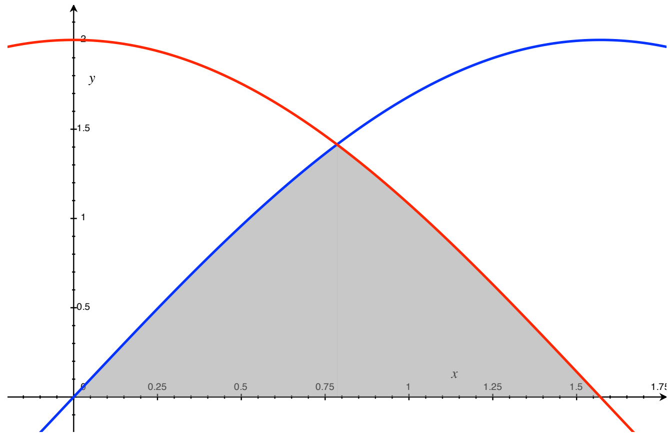
Math 2260 Written HW #2 Solutions

1. Find the area of the region that is enclosed between the curves $y = 2\sin(x)$ and $y = 2\cos(x)$ from $x = 0$ to $x = \pi/2$.

   Answer: The desired area is pictured below:



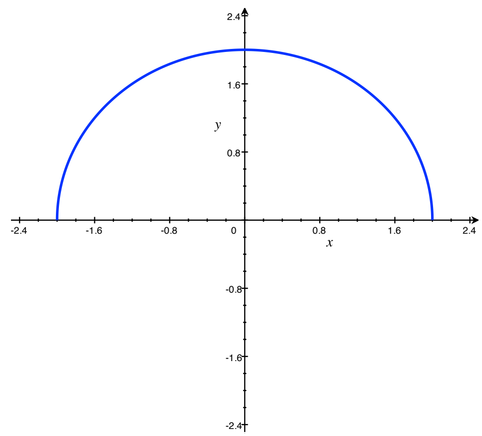
In the figure, the blue curve is $y = 2\sin(x)$ and the red curve is $y = 2\cos(x)$. Since the top of the region is the blue curve between 0 and $\pi/4$ (since $2\cos(x) = 2\sin(x)$ when $x = \pi/4$), whereas the top of the region is the red curve between $\pi/4$ and $\pi/2$, we need to compute the area in two parts:

$$\begin{aligned}
\text{Area} &= \int_0^{\pi/4} 2\sin(x)\,dx + \int_{\pi/4}^{\pi/2} 2\cos(x)\,dx \\
&= \Big[-2\cos(x)\Big]_0^{\pi/4} + \Big[2\sin(x)\Big]_{\pi/4}^{\pi/2} \\
&= [-2\cos(\pi/4) + 2\cos(0)] + [2\sin(\pi/2) - 2\sin(\pi/4)] \\
&= \left[-2\cdot\frac{1}{\sqrt{2}} + 2\cdot 1\right] + \left[2\cdot 1 - 2\cdot\frac{1}{\sqrt{2}}\right] \\
&= \left[-\sqrt{2} + 2\right] + \left[2 - \sqrt{2}\right] \\
&= 4 - 2\sqrt{2}.
\end{aligned}$$

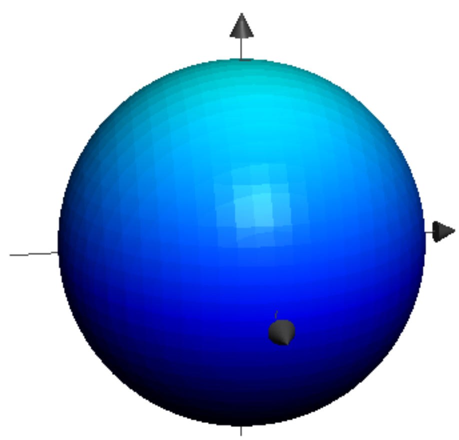
Therefore, the area of the shaded region is $4 - 2\sqrt{2}$.

2. Find the volume of the solid generated by revolving the region bounded by the x-axis and the semicircle $y = \sqrt{4 - x^2}$ around the x-axis.

Answer: First, here's the graph of the function $y = \sqrt{4 - x^2}$:



Revolving around the x-axis will yield a sphere like this one:



Now, each cross section of the sphere is a circle of radius $\sqrt{4 - x^2}$, so we know that the cross-sectional area is

$$A(x) = \pi \left(\sqrt{4 - x^2} \right)^2 = \pi(4 - x^2) = 4\pi - \pi x^2.$$

Now the volume of this solid is given by integrating $A(x)$ along its length. Since the graph $y = \sqrt{4 - x^2}$ intersects the x-axis at $x = \pm 2$, we should integrate from -2 to 2. Therefore, the volume of the sphere is

$$\begin{aligned}
\int_{-2}^{2} A(x)\,dx &= \int_{-2}^{2} \left(4\pi - \pi x^2\right)\,dx \\
&= \left[4\pi x - \pi \frac{x^3}{3}\right]_{-2}^{2} \\
&= \left(4\pi(2) - \pi \cdot \frac{2^3}{3}\right) - \left(4\pi(-2) - \pi \cdot \frac{(-2)^3}{3}\right) \\
&= \left(8\pi - \frac{8\pi}{3}\right) - \left(-8\pi + \frac{8\pi}{3}\right) \\
&= 16\pi - \frac{16\pi}{3} \\
&= \frac{32\pi}{3}.
\end{aligned}$$

Thus, the volume of the given solid is $\frac{32\pi}{3}$.