Math 2260 Written HW #14 Solutions

1. (a) Describe geometrically the set of points (x, y, z) satisfying the inequalities

$$1 \leq x^2 + y^2 + z^2 \leq 4.$$

Answer: First of all, notice that the points (x, y, z) satisfying $x^2 + y^2 + z^2 = 1$ are the points on the sphere of radius 1 around the origin. Therefore, the points satisfying $x^2 + y^2 + z^2 \geq 1$ are those points either on the sphere of radius 1 or outside of it.

Likewise, the points satisfying the inequality $x^2 + y^2 + z^2 \leq 4$ are those points lying either on or inside the sphere of radius 2 around the origin.

Therefore, the points satisfying the inequalities $1 \leq x^2 + y^2 + z^2 \leq 4$ are those points lying either between the sphere of radius 1 and the sphere of radius 2 or on one of these two sphere. In other words, these inequalities describe a solid ball with the middle scooped out.

(b) Describe geometrically the set of points (x, y, z) satisfying the inequalities

$$x^2 + y^2 + z^2 \leq 1 \quad \text{and} \quad z \geq 0.$$

Answer: If a point (x, y, z) satisfies the inequality $x^2 + y^2 + z^2 \leq 1$, then the point lies either on the sphere of radius 1 centered at the origin or inside it. On the other hand, if $z \geq 0$ then the point must lie on or above the xy-plane.

Therefore, a point satisfying both $x^2 + y^2 + z^2 \leq 1$ and $z \geq 0$ must lie both inside the sphere and above the plane, so this is the top half of the solid ball of radius 1.

You can describe these sets purely in terms of words, but it may be helpful to draw a picture.

2. Let $\vec{\mathbf{u}} = 2\vec{\imath} + 3\vec{\jmath}$, $\vec{\mathbf{v}} = 8\vec{\imath} - 13\vec{\jmath}$, and $\vec{\mathbf{w}} = -\vec{\imath} + \vec{\jmath}$. Find real numbers a and b so that

$$\vec{\mathbf{u}} = a\vec{\mathbf{v}} + b\vec{\mathbf{w}}.$$

Answer: If $\vec{\mathbf{u}} = a\vec{\mathbf{v}} + b\vec{\mathbf{w}}$, then we have that

$$\begin{aligned} 2\vec{\imath} + 3\vec{\jmath} &= a(8\vec{\imath} - 13\vec{\jmath}) + b(-\vec{\imath} + \vec{\jmath}) \\ 2\vec{\imath} + 3\vec{\jmath} &= (8a - b)\vec{\imath} + (-13a + b)\vec{\jmath} \end{aligned}$$

Since the components must match, this gives us the system of equations

$$\begin{aligned} 2 &= 8a - b \\ 3 &= -13a + b \end{aligned}$$

From the first equation, then, $b = 8a - 2$. Plugging this into the second equation yields

$$3 = -13a + (8a - 2) = -5a - 2.$$

Hence,

$$5a = -5,$$

so $a = -1$ and, hence, $b = 8a - 2 = 8(-1) - 2 = -10$.

I can double-check that $a = -1$ and $b = -10$ are right:

$$a\vec{\mathbf{v}} + b\vec{\mathbf{w}} = -1(8\vec{\imath} - 13\vec{\jmath}) - 10(-\vec{\imath} + \vec{\jmath}) = (-8\vec{\imath} + 13\vec{\jmath}) + (10\vec{\imath} - 10\vec{\jmath}) = 2\vec{\imath} + 3\vec{\jmath},$$

which is indeed equal to $\vec{\mathbf{u}}$.