Math 2260 Written HW #11 Solutions

1. Does the series
$$\sum_{k=0}^{\infty} \frac{\cos(k\pi)}{5^k}$$
converge or diverge? If it converges, find its sum, and if it diverges, explain why.

Answer: Notice that $\cos(k\pi)$ is equal to 1 if k is even and -1 if k is odd; in other words,
$$\cos(k\pi) = (-1)^k$$
for all k. Therefore,
$$\sum_{k=0}^{\infty} \frac{\cos(k\pi)}{5^k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{5^k} = \sum_{k=0}^{\infty} \left(\frac{-1}{5}\right)^k,$$
which can also be written as
$$\sum_{k=1}^{\infty} \left(\frac{-1}{5}\right)^{k-1}.$$
Either way, this is a geometric series with $a = 1$ and $r = -1/5$. Since $|-1/5| < 1$, we know the series converges to
$$\frac{1}{1-(-1/5)} = \frac{1}{6/5} = \frac{5}{6}.$$

2. Find examples of convergent geometric series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ so that

- $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$ for some numbers A and B,
- $\sum_{k=1}^{\infty} a_k b_k$ converges, but
- $\sum_{k=1}^{\infty} a_k b_k \neq AB$.

Answer: Let $a_k = \left(\frac{-1}{2}\right)^{k-1}$ and let $b_k = \left(\frac{-1}{3}\right)^{k-1}$ for all k. Then
$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \left(\frac{-1}{2}\right)^{k-1} = \frac{1}{1-(-1/2)} = \frac{1}{3/2} = \frac{2}{3}$$
since this is a geometric series. Similarly,
$$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \left(\frac{-1}{3}\right)^{k-1} = \frac{1}{1-(-1/3)} = \frac{1}{4/3} = \frac{3}{4}.$$
On the other hand,
$$\sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{\infty} \left(\frac{-1}{2}\right)^{k-1} \left(\frac{-1}{3}\right)^{k-1} = \sum_{k=1}^{\infty} \left(\frac{1}{6}\right)^{k-1} = \frac{1}{1-1/6} = \frac{1}{5/6} = \frac{6}{5},$$
which is not equal to $\frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2}$, so these series satisfy the conditions laid out above.

3. Does the series
$$\sum_{k=1}^{\infty} \frac{e^k}{1+e^{2k}}$$
converge or diverge? Explain your answer.

Answer: I intend to show that the series converges by using the Integral Test. Hence, I need to evaluate the integral
$$\int_1^\infty \frac{e^x}{1+e^{2x}}\,dx = \lim_{b\to\infty} \int_1^b \frac{e^x}{1+e^{2x}}\,dx.$$
Let $u = e^x$. Then $du = e^x\,dx$ and, since $u(1) = e^1 = e$ and $u(b) = e^b$, the above integral becomes
$$\lim_{b\to\infty} \int_e^{e^b} \frac{du}{1+u^2}.$$
In turn, using the trig substitution $u = \tan\theta$, I have $du = \sec^2\theta\,d\theta$ and hence the above integral is equal to
$$\begin{aligned}
\lim_{b\to\infty} \int_{\arctan(e)}^{\arctan(e^b)} \frac{\sec^2\theta\,d\theta}{1+\tan^2\theta} &= \lim_{b\to\infty} \int_{\arctan(e)}^{\arctan(e^b)} \frac{\sec^2\theta\,d\theta}{\sec^2\theta} \\
&= \lim_{b\to\infty} \int_{\arctan(e)}^{\arctan(e^b)} d\theta \\
&= \lim_{b\to\infty} \Big[\theta\Big]_{\arctan(e)}^{\arctan(e^b)} \\
&= \lim_{b\to\infty} \Big(\arctan(e^b) - \arctan(e)\Big) \\
&= \frac{\pi}{2} - \arctan(e).
\end{aligned}$$
This is some number (which is approximately 0.353...) so the integral converges. Therefore, the Integral Test implies that the series $\sum_{k=1}^{\infty} \frac{e^k}{1+e^{2k}}$ also converges.