Math 2250 Final Exam Practice Problem Solutions

1. What are the domain and range of the function

$$f(x) = \frac{\ln x}{\sqrt{x}}?$$

Answer: $\sqrt{x}$ is only defined for $x \geq 0$, and $\ln x$ is only defined for $x > 0$. Hence, the domain of the function is $x > 0$. Notice that

$$\lim_{x \to 0^+} \frac{\ln x}{\sqrt{x}} = -\infty,$$

since $\sqrt{x} \to 0^+$ as $x \to 0^+$. Now, we can evaluate

$$\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}}$$

using L'Hôpital's Rule; it is equal to

$$\lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x \to \infty} \frac{2\sqrt{x}}{x} = \lim_{x \to \infty} \frac{2}{\sqrt{x}} = 0.$$

Therefore, f will have some maximum value; to figure out what it is, take

$$f'(x) = \frac{\sqrt{x}\frac{1}{x} - \ln x \frac{1}{2\sqrt{x}}}{x} = \frac{\frac{2}{2\sqrt{x}} - \frac{\ln x}{2\sqrt{x}}}{x} = \frac{2 - \ln x}{2x^{3/2}}.$$

Then $f'(x) = 0$ when

$$0 = 2 - \ln x,$$

meaning that $\ln x = 2$, or $x = e^2$. Notice that $f'(x)$ changes sign from positive to negative at $x = e^2$, so the maximum of f occurs here. Since

$$f(e^2) = \frac{\ln e^2}{\sqrt{e^2}} = \frac{2}{e},$$

we see that the range of f is

$$\left(-\infty, \frac{2}{e}\right].$$

2. Find the point on the graph of $y = e^{3x}$ at which the tangent line passes through the origin.

Answer: Let $f(x) = e^{3x}$. Since

$$f'(x) = 3e^{3x},$$

the tangent line to e^{3x} at the point $x = a$ has slope $3e^{3a}$; hence, using the point-slope formula, it is given by

$$y - e^{3a} = 3e^{3a}(x - a) = 3e^{3a}x - 3ae^{3a}.$$

In other words, the tangent line to the curve at $x = a$ is

$$y = 3e^{3a}x - 3ae^{3a} + e^{3a}$$

or

$$y = e^{3a}(3x - 3a + 1).$$

This passes through the origin if we get equality when we substitute 0 for both x and y, so it must be the case that

$$0 = e^{3a}(0 - 3a + 1) = e^{3a}(1 - 3a).$$

Since $e^{3a} \neq 0$, this means that $1 - 3a = 0$, or $a = 1/3$. Therefore, since

$$f(1/3) = e^{3 \cdot 1/3} = e,$$

the point whose tangent line passes through the origin is

$$\left(\frac{1}{3}, e\right).$$

3. Find the equation of the tangent line to the curve

$$xy^3 - x^2 y = 6$$

at the point $(3, 2)$.

Answer: Differentiating both sides with respect to x yields

$$y^3 + 3xy^2 \frac{dy}{dx} - 2xy - x^2 \frac{dy}{dx} = 0.$$

Therefore,

$$\frac{dy}{dx}\left(3xy^2 - x^2\right) = 2xy - y^3.$$

Thus,

$$\frac{dy}{dx} = \frac{2xy - y^3}{3xy^2 - x^2}.$$

Plugging in $(3, 2)$, we see that the slope of the tangent line is

$$\frac{2(3)(2) - 2^3}{3(3)(2)^2 - 3^2} = \frac{12 - 8}{36 - 9} = \frac{4}{27}.$$

Thus, using the point-slope formula, the equation of the tangent line is

$$y - 2 = \frac{4}{27}(x - 3) = \frac{4}{27}x - \frac{12}{27},$$

or, equivalently,

$$y = \frac{4}{27}x + \frac{14}{9}.$$

4. Use an appropriate linearization to approximate $\sqrt{96}$.

Answer: Let $f(x) = \sqrt{x}$. Then I will approximate $\sqrt{96}$ using the linearization of f at $a = 100$. To do so, first take

$$f'(x) = \frac{1}{2\sqrt{x}}.$$

Then the linearization is

$$L(x) = f(100) + f'(100)(x - 100) = 10 + \frac{1}{20}(x - 100) = 10 + \frac{x}{20} - 5 = \frac{x}{20} + 5.$$

Therefore,

$$\sqrt{96} = f(96) \approx L(96) = 5 + \frac{96}{20} = 5 + \frac{48}{10} = \frac{98}{10} = 9.8.$$

So we approximate $\sqrt{96}$ by 9.8.

5. Consider the function $f(x) = x^2 e^{-x^2}$. What is the absolute maximum of $f(x)$?

   Answer: Notice that f is defined for all x. Also,

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{x^2}{e^{x^2}} = 0,$$

   (by two applications of L'Hôpital's Rule) so f doesn't go off to infinity.

   Now, to find the critical points, compute

$$f'(x) = 2xe^{-x^2} + x^2 e^{-x^2}(-2x) = e^{-x^2}(2x - 2x^3),$$

   which equals zero precisely when $0 = 2x - 2x^3 = 2x(1 - x^2)$; namely when $x = 0$ or $x = \pm 1$ Thus, we just need to evaluate f at the critical points:

$$f(1) = 1/e$$
$$f(0) = 0$$
$$f(-1) = 1/e$$

   Since f limits to 0 in both directions, we see that the absolute maximum value of the function (occurring at both $x = 1$ and $x = -1$) is $1/e$.

6. Water is draining from a conical tank at the rate of 18 cubic feet per minute. The tank has a height of 10 feet and the radius at the top is 5 feet. How fast (in feet per minute) is the water level changing when the depth is 6 feet? (Note: the volume of a cone of radius r and height h is $\frac{\pi r^2 h}{3}$.)

   Answer: If h is the height of the top of the water in the cone and r is the radius of the top of the water, then

$$\frac{r}{5} = \frac{h}{10},$$

   so $r = h/2$. Now, the volume of water in the tank is

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi (h/2)^2 h = \frac{\pi}{12} h^3.$$

   In turn, this means that

$$\frac{dV}{dt} = \frac{\pi}{12} 3h^2 \frac{dh}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}.$$

   Since $\frac{dV}{dt} = 18$, this means that

$$18 = \frac{\pi}{4} h^2 \frac{dh}{dt},$$

   or

$$\frac{dh}{dt} = \frac{72}{\pi h^2}.$$

   Thus, when $h = 6$, the water level is changing at the rate

$$\frac{dh}{dt} = \frac{72}{36\pi} = \frac{2}{\pi}.$$

7. The function $f(x) = x^4 - 6x^3$ is concave down for what values of x?

   Answer: To determine concavity, we need to compute the second derivative. Now,

$$f'(x) = 4x^3 - 18x^2,$$

   so

$$f''(x) = 12x^2 - 36x = 12x(x - 3).$$

   Notice that $f''(x) < 0$ precisely when $0 < x < 3$, so the function f is concave down on the interval $(0, 3)$.

8. Evaluate the limit

$$\lim_{x\to 0}(1-6x)^{1/x}.$$

Answer: Let $f(x)-(1-6x)^{1/x}$. Taking the natural log of f yields

$$\ln((1-6x)^{1/x}) = \frac{1}{x}\ln(1-6x) = \frac{\ln(1-6x)}{x}.$$

Now, by L'Hôpital's Rule,

$$\lim_{x\to 0}\ln(f(x)) = \lim_{x\to 0}\frac{\ln(1-6x)}{x} = \lim_{x\to 0}\frac{\frac{-6}{1-6x}}{1} = \lim_{x\to 0}\frac{-6}{1-6x} = -6.$$

Therefore, since this is the limit of $\ln(f(x))$, we know that

$$\lim_{x\to 0} f(x) = e^{-6}.$$

9. Let $f(x) = x^{\cos x}$. What is $f'(\pi/2)$?

Answer: I will use logarithmic differentiation to find $f'(x)$. To that end, let $y = f(x) = x^{\cos x}$. Then

$$\ln y = \ln(x^{\cos x}) = \cos x \ln x.$$

Differentiating both sides,

$$\frac{1}{y}\frac{dy}{dx} = \cos x\frac{1}{x} - \sin x \ln x = \frac{\cos x}{x} - \sin x \ln x.$$

Therefore,

$$f'(x) = \frac{dy}{dx} = y\left(\frac{\cos x}{x} - \sin x\ln x\right) = x^{\cos x}\left(\frac{\cos x}{x} - \sin x \ln x\right).$$

Hence,

$$f'(\pi/2) = (\pi/2)^{\cos\pi/2}\left(\frac{\cos\pi/2}{\pi/2} - \sin\pi/2\ln(\pi/2)\right) = (\pi/2)^0(0 - 1\cdot\ln(\pi/2)) = -\ln(\pi/2).$$

10. For $0 \leq t \leq 5$, a particle moves in a horizontal line with acceleration $a(t) = 2t-4$ and initial velocity $v(0) = 3$.

   (a) When is the particle moving to the left?

   Answer: The particle will be moving to the left when its velocity is negative. To determine the velocity, note that

$$\int a(t)dt = \int(2t-4)dt = t^2 - 4t + C.$$

   Hence, $v(t) = t^2 - 4t + C$ for some C, which we can determine by plugging in $t = 0$:

$$3 = v(0) = 0^2 - 4(0) + C = C,$$

   so $v(t) = t^2 - 4t + 3 = (t-3)(t-1)$. Notice that this function is negative when $1 < t < 3$, so the particle is moving to the left between $t = 1$ and $t = 3$.

   (b) When is the particle speeding up?

   Answer: The particle is speeding up when its acceleration is positive, which is to say when

$$0 < a(t) = 2t - 4,$$

   so the particle is speeding up when $t > 2$.

(c) What is the position of the particle at time t if the initial position of the particle is 6?
Answer: Since
$$\int v(t)dt = \int (t^2 - 4t + 3)dt = \frac{t^3}{3} - 2t^2 + 3t + D,$$
we know that $s(t) = \frac{t^3}{3} - 2t^2 + 3t + D$ for some real number D, which we can solve for by plugging in $t = 0$:
$$6 = s(0) = \frac{0^3}{3} - 2(0)^2 + 3(0) + D = D,$$
so the position of the particle at time t is
$$s(t) = \frac{t^3}{3} - 2t^2 + 3t + 6.$$

11. If $\int_0^6 f(x)dx = 10$ and $\int_0^4 f(x)dx = 7$, find $\int_4^6 f(x)dx$.
Answer: Notice that
$$\int_4^6 f(x)dx = \int_0^6 f(x)dx - \int_0^4 f(x)dx = 10 - 7 = 3.$$

12. Evaluate the definite integral
$$\int_{\pi/6}^{\pi/4} \sin t dt.$$
Answer: Since $-\cos t$ is an antiderivative of $\sin t$, the Fundamental Theorem of Calculus tells us that
$$\int_{\pi/6}^{\pi/4} \sin t dt = \Big[- \cos t \Big]_{\pi/6}^{\pi/4} = -\cos(\pi/4) - (-\cos(\pi/6)) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} = \frac{\sqrt{3} - \sqrt{2}}{2}.$$

13. Evaluate the integral
$$\int \frac{2}{t-3} dt.$$
Answer: Since $\frac{2}{t-3}$ looks vaguely like $\frac{1}{t}$, we should expect that the natural log comes into play. In fact, $2\ln(t-3)$ is an antiderivative of $\frac{2}{t-3}$, so
$$\int \frac{2}{t-3} dt = 2\ln(t-3) + C.$$

14. Evaluate the definite integral
$$\int_1^4 \frac{2\sqrt{x} + 4x^2}{x} dx$$
Answer: Re-write the integral as
$$\int_1^4 \left(\frac{2\sqrt{x}}{x} + \frac{4x^2}{x} \right) dx = \int_1^4 \frac{2\sqrt{x}}{x} dx + \int_1^4 \frac{4x^2}{x} dx = \int_1^4 \frac{2}{\sqrt{x}} dx + \int_1^4 4x dx.$$
Now,
$$\int_1^4 \frac{2}{\sqrt{x}} dx = \int_1^4 2x^{-1/2} dx = \left[\frac{2x^{1/2}}{1/2} \right]_1^4 = \left[4\sqrt{x} \right]_1^4 = 8 - 4 = 4.$$

On the other hand,

$$\int_1^4 4x dx = \left[2x^2\right]_1^4 = 32 - 2 = 30.$$

Therefore,

$$\int_1^4 \frac{2\sqrt{x} + 4x^2}{x} dx = \int_1^4 \frac{2}{\sqrt{x}} dx + \int_1^4 4x dx = 4 + 30 = 34.$$

15. Suppose the velocity of a particle is given by

$$v(t) = 6t^2 - 4t.$$

What is the displacement of the particle from 0 to 2?

Answer: The displacement is given by

$$s(2) - s(0).$$

Since $s'(t) = v(t)$, the Fundamental Theorem tells us that

$$s(2) - s(0) = \int_0^2 s'(t) dt = \int_0^2 v(t) dt = \int_0^2 (6t^2 - 4t) dt = \left[2t^3 - 2t^2\right]_0^2 = (16 - 8) - (0 - 0) = 8.$$

Therefore, the displacement is 8 units.

16. Suppose that

$$\int_0^{x^2} f(t) dt = \sqrt{x^2 + 1} - 1.$$

What is $f(2)$?

Answer: Let $g(x) = \sqrt{x^2 + 1} - 1$. Then,

$$g'(x) = \frac{d}{dx}\left(\int_0^{x^2} f(t) dt\right) = \frac{d}{du}\left(\int_0^{u} f(t) dt\right)\frac{du}{dx}$$

where $u = x^2$, using the Chain Rule.

Therefore, by the first part of the Fundamental Theorem,

$$g'(x) = f(u) \cdot 2x = 2x f(x^2).$$

In other words,

$$f(x^2) = \frac{g'(x)}{2x}.$$

Now, we know that $g(x) = \sqrt{x^2 + 1} - 1$, so

$$g'(x) = \frac{1}{2\sqrt{x^2 + 1}}(2x) = \frac{x}{\sqrt{x^2 + 1}}.$$

Hence,

$$f(2) = \frac{g'(\sqrt{2})}{2\sqrt{2}} = \frac{\frac{\sqrt{2}}{\sqrt{3}}}{2\sqrt{2}} = \frac{1}{2\sqrt{3}}.$$

17. Evaluate the integral

$$\int 3e^{\tan x} \sec^2 x \, dx.$$

Answer: Let $u = \tan x$. Then $du = \sec^2 x \, dx$, so we can re-write the above integral as

$$\int 3e^u \, du = 3e^u + C.$$

Now, substituting back in for u, this yields the answer

$$3e^{\tan x} + C.$$

18. Evaluate the definite integral

$$\int_0^{\pi/16} 8 \tan(4x) \, dx.$$

Answer: Notice that $\tan(4x) = \frac{\sin(4x)}{\cos(4x)}$, so the above integral is equal to

$$\int_0^{\pi/16} 8 \frac{\sin(4x)}{\cos(4x)} \, dx.$$

Let $u = \cos(4x)$. Then $du = -4 \sin(4x) \, dx$. Therefore,

$$8 \frac{\sin(4x)}{\cos(4x)} \, dx = \frac{-2}{\cos(4x)} \cdot (-4 \sin(4x) dx) = \frac{-2}{u} \, du.$$

We want to replace the given integral with an integral in terms of u, but that means we also need to change the limits of integration:

$$u(0) = \cos(4 \cdot 0) = \cos(0) = 1 \quad \text{and} \quad u(\pi/16) = \cos(4 \cdot \pi/16) = \cos(\pi/4) = \frac{1}{\sqrt{2}}.$$

Therefore, the integral we were given can be re-written as

$$\int_1^{1/\sqrt{2}} \frac{-2}{u} \, du = [-2 \ln u]_1^{1/\sqrt{2}} = -2 \ln(1/\sqrt{2}) + 2 \ln(1) = -2 \ln(1/\sqrt{2}) = -2 \ln\left(2^{-1/2}\right).$$

But now, from the properties of logarithms,

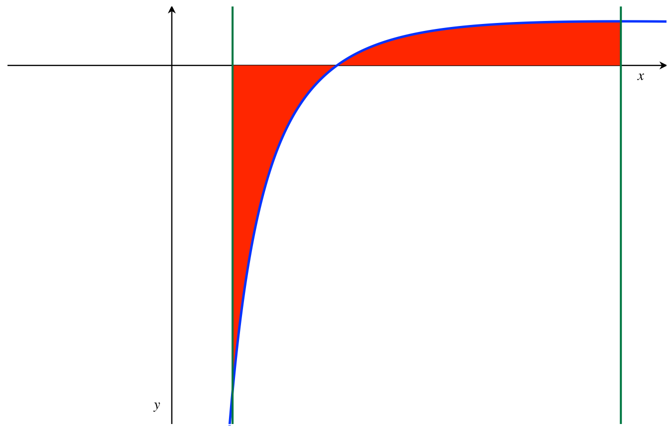
$$-2 \ln\left(2^{-1/2}\right) = \ln\left(\left(2^{-1/2}\right)^{-2}\right) = \ln\left(2^1\right) = \ln 2.$$

Hence, we conclude that $\int_0^{\pi/16} 8 \tan(4x) \, dx = \ln 2$.

19. What is the area of the red region in the figure? The blue curve is given by $y = 2\frac{\ln x}{x}$ and the vertical green lines are the lines $x = \frac{1}{e}$ and $x = e$.

Answer: First, notice that the curve $y = 2\frac{\ln x}{x}$ crosses the x-axis when

$$2\frac{\ln x}{x} = 0,$$



which only occurs when $\ln x = 0$, meaning when $x = 1$. Between $x = 1/e$ and $x = 1$, the x-axis (i.e. the curve $y = 0$) is above the blue curve, whereas the blue curve is above the x-axis when $1 < x \leq e$. Therefore, the red area is equal to

$$\int_{1/e}^{1} \left(0 - 2\frac{\ln x}{x}\right) dx + \int_{1}^{e} \left(2\frac{\ln x}{x} - 0\right) dx = \int_{1/e}^{1} -2\frac{\ln x}{x}\, dx + \int_{1}^{e} 2\frac{\ln x}{x}\, dx.$$

For both of these integrals, let $u = \ln x$. Then $du = \frac{1}{x}\, dx$. Moreover,

$$\begin{aligned} u(1/e) &= \ln(1/e) = \ln(e^{-1}) = -1 \\ u(1) &= \ln(1) = \ln(e^0) = 0 \\ u(e) &= \ln(e) = \ln(e^1) = 1. \end{aligned}$$

Therefore, the sum of integrals above is equal to

$$\begin{aligned} \int_{-1}^{0} -2u\, du + \int_{0}^{1} 2u\, du &= \left[-u^2\right]_{-1}^{0} + \left[u^2\right]_{0}^{1} \\ &= \left(-(0)^2 + (-1)^2\right) + \left(1^2 - 0^2\right) \\ &= 2, \end{aligned}$$

so the red region has area 2.

20. What are the domain and range of the function

$$f(x) = \frac{1 + e^x}{1 - e^x}?$$

Answer: The function is well-defined everywhere except when the denominator is zero, which happens when

$$0 = 1 - e^x,$$

or, equivalently, $e^x = 0$. This only happens when $x = 0$, so we see that the domain of f is the set of all real numbers $\neq 0$.

As for the range of f, notice that

$$\lim_{x \to 0^-} \frac{1 + e^x}{1 - e^x} = +\infty$$

and

$$\lim_{x\to 0^+} \frac{1+e^x}{1-e^x} = -\infty.$$

Now,

$$\lim_{x\to -\infty} \frac{1+e^x}{1-e^x} = 1$$

and

$$\lim_{x\to +\infty} \frac{1+e^x}{1-e^x} = \lim_{x\to +\infty} \frac{e^x}{-e^x} = -1$$

by L'Hôpital's Rule, so f has horizontal asymptotes at 1 and -1. Finally, since

$$f'(x) = \frac{(1-e^x)e^x - (1+e^x)(-e^x)}{(1-e^x)^2} = \frac{e^x - e^{2x} + e^x + e^{2x}}{(1-e^x)^2} = \frac{2e^x}{(1-e^x)^2}$$

is never zero, f has no critical points and, thus, no maxima or minima. Putting this all together, the range of f consists of those real numbers x such that

$$-\infty < x < -1 \quad \text{or} \quad 1 < x < +\infty.$$

21. What is the equation of the line tangent to the graph of $y^3 + 3x^2y^2 + 2x^3 = 4$ at the point $(1, -1)$?

Answer: The goal is to determine y' by using implicit differentiation.

Differentiating both sides yields:

$$3y^2 \cdot y' + 6xy^2 + 3x^2 \cdot 2y \cdot y' + 6x^2 = 0,$$

or, equivalently,

$$y'(3y^2 + 6x^2y) + 6xy^2 + 6x^2 = 0.$$

Hence,

$$y' = \frac{-6xy^2 - 6x^2)}{3y^2 + 6x^2y}.$$

Therefore, the slope of the tangent line at $(1, -1)$ will be

$$\frac{-6(1)(-1)^2 - 6(1)^2}{3(-1)^2 + 6(1)^2(-1)} = \frac{-12}{-3} = 4.$$

Hence, using the point-slope formula, the tangent line at $(1, -1)$ will be

$$y + 1 = 4(x-1) = 4x - 4,$$

or, equivalently,

$$y = 4x - 5.$$

22. Evaluate the limit

$$\lim_{x\to 0} \frac{1-\cos x}{x^2 + x}.$$

Answer: Notice that both numerator and denominator go to zero as $x \to 0$. Hence, we can apply L'Hopital's Rule:

$$\lim_{x\to 0} \frac{1-\cos x}{x^2 + x} = \lim_{x\to 0} \frac{\sin x}{2x+1} = 0,$$

since $\sin(0) = 0$.

23. The volume of a cube is increasing at a rate of 10 cm^3/min. How fast is the surface area increasing when the length of an edge is 10 cm?

Answer: If $V(t)$ is the volume of the cube after t minutes, $A(t)$ is the surface area, and $\ell(t)$ is the length of an edge, then we know that

$$
\begin{aligned}
V(t) &= (s(t))^3 \\
A(t) &= 6(s(t))^2 \\
V'(t) &= 10 \text{ for all } t \\
s(t_0) &= 10 \text{ where } t_0 is the moment of interest
\end{aligned}
$$

and we're asked to figure out $A'(t_0)$. Now, we know that

$$A'(t) = 6 \cdot 2s(t) \cdot s'(t) = 12s(t)s'(t),$$

so

$$A'(t_0) = 12s(t_0)s'(t_0) = 12(10)s'(t_0) = 120s'(t_0),$$

so all we need to do is determine $s'(t_0)$. To do so, let's differentiate V:

$$V'(t) = 3(s(t))^2 \cdot s'(t).$$

Then, plugging in $t = t_0$, we have

$$10 = V'(t_0) = 3(s(t_0))^2 \cdot s'(t_0) = 3(10)^2 \cdot s'(t_0) = 300s'(t_0),$$

meaning that $s'(t_0) = \frac{10}{300} = \frac{1}{30}$. Therefore, we know that

$$A'(t_0) = 120s'(t_0) = 120 \cdot \frac{1}{30} = 4,$$

so the surface area is increasing at a rate of 4 cm^2/sec.

24. Find the maximum and minimum values, inflection points and asymptotes of $y = \ln(x^2 + 1)$ and use this information to sketch the graph.

Answer: Notice that

$$y' = \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1}$$

and, by the Quotient Rule,

$$y'' = \frac{(x^2+1)(2) - 2x(2x)}{(x^2+1)^2} = \frac{2x^2+2-4x^2}{(x^2+1)^2} = \frac{2-2x^2}{(x^2+1)^2}.$$

Now, the critical points occur when $y' = 0$, which is to say when

$$\frac{2x}{x^2+1} = 0.$$

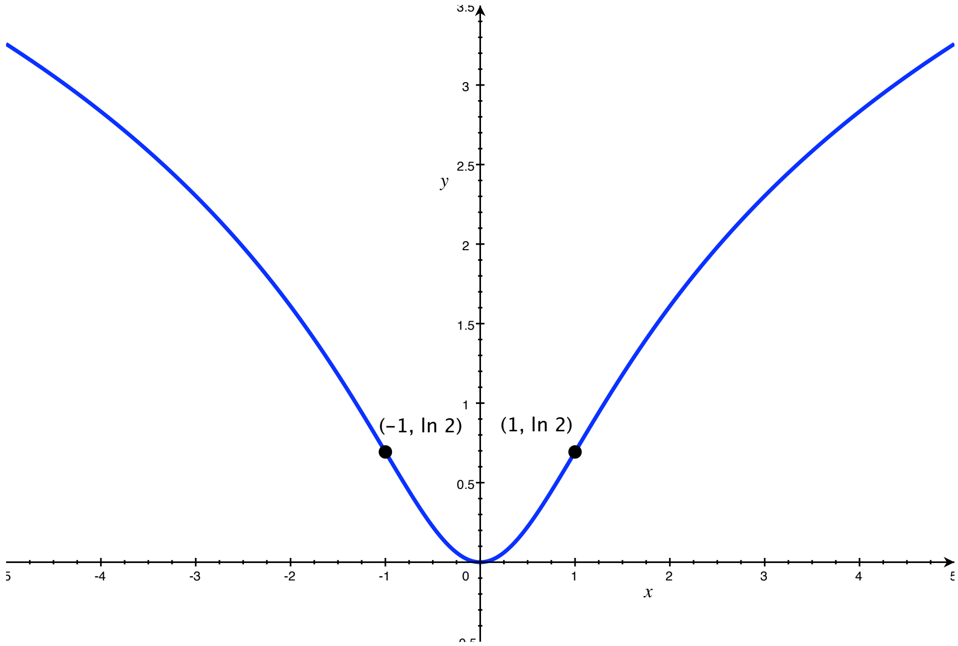
The only happens when $x = 0$, so 0 is the only critical point. Notice that $y''(0) = 2$, which is greater than zero, so the second derivative test implies that 0 is a local minimum.

$y'' = 0$ when $2 - 2x^2 = 0$, meaning when $x = \pm 1$, so there are inflection points at $x = \pm 1$. Finally,

$$\lim_{x \to -\infty} \ln(x^2+1) = \infty = \lim_{x \to +\infty} \ln(x^2+1),$$

so there are no horizontal asymptotes.

Putting all this together, we see that y has a minimum at 0 and is concave up between -1 and 1 and concave down everywhere else and has no asymptotes, meaning that the graph looks something like this:



25. Use an appropriate linearization to approximate $e^{1/10}$.

Answer: Consider the function $f(x) = e^x$. Then I want to approximate $f(1/10) = e^{1/10}$ using the linearization of f at $x = 0$. This linearization is given by

$$L(x) = f(0) + f'(0)(x - 0) = f(0) + f'(0)x.$$

Now $f(0) = e^0 = 1$ and $f'(x) = e^x$, so $f'(0) = e^0 = 1$. Hence, the linearization is given by

$$L(x) = 1 + x.$$

Therefore,

$$f(1/10) \approx L(1/10) = 1 + \frac{1}{10} = \frac{11}{10} = 1.1.$$

26. What is the absolute maximum value of $f(x) = x^{1/x}$ for $x > 0$?

Answer: Taking the natural log of both sides,

$$\ln f(x) = \ln(x^{1/x}) = \frac{1}{x}\ln x$$

. Now differentiating, we see that

$$\frac{f'(x)}{f(x)} = \frac{1}{x} \cdot \frac{1}{x} - \frac{1}{x^2}\ln x = \frac{1}{x^2}(1 - \ln x),$$

so

$$f'(x) = f(x)\frac{1}{x^2}(1 - \ln x) = \frac{x^{1/x}}{x^2}(1 - \ln x).$$

Since $x^{1/x}$ is never zero for $x > 0$, $f'(x) = 0$ only when $1 - \ln x = 0$, meaning that $\ln x = 1$. This only happens when $x = e$, so e is the only critical point of f. Notice that $f'(x)$ changes sign from positive to negative at $x = e$, so the first derivative test implies that f has a local maximum at e. However, since this is the only critical point and there are no endpoints, this must, in fact, be the global maximum of f.

Hence, the absolute maximum value of $f(x)$ for $x > 0$ is

$$f(e) = e^{1/e}.$$

27. Suppose the velocity of a particle is given by

$$v(t) = 3\cos t + 4\sin t.$$

If the particle starts (at time 0) at a position 7 units to the right of the origin, what is the position of the particle at time t?

Answer: Let $s(t)$ be the position of the particle at time t. Then we know that $s'(t) = v(t)$ and that $s(0) = 7$. Now,

$$\int v(t)dt = \int (3\cos t + 4\sin t)dt = 3\sin t - 4\cos t + C.$$

Therefore, since $s(t)$ is *an* antiderivative of $v(t) = s'(t)$, we know that

$$s(t) = 3\sin t - 4\cos t + C$$

for some real number C. To solve for C, plug in $t = 0$:

$$7 = s(0) = 3\sin(0) - 4\cos(0) + C = -4 + C,$$

so we see that $C = 11$.

Therefore, the position of the particle is given by

$$s(t) = 3\sin t - 4\cos t + 11.$$

28. Evaluate the definite integral

$$\int_0^{\pi/6} \frac{2 + \cos^3\theta}{\cos^2\theta} d\theta.$$

Answer: Note that

$$\frac{2 + \cos^3\theta}{\cos^2\theta} = \frac{2}{\cos^2\theta} + \cos\theta = 2\sec^2\theta + \cos\theta.$$

Therefore,

$$\int_0^{\pi/6} \frac{2 + \cos^3\theta}{\cos^2\theta} d\theta = \int_0^{\pi/6} (2\sec^2\theta + \cos\theta) d\theta.$$

By the Fundamental Theorem of Calculus, this is equal to

$$[2\tan\theta + \sin\theta]_0^{\pi/6} = (2\tan(\pi/6) + \sin(\pi/6)) - (2\tan(0) + \sin(0)) = \frac{2}{\sqrt{3}} + \frac{1}{2}.$$

29. Evaluate

$$\int \csc r \cot r \, dr.$$

Answer: Recall that the derivative of $\csc r$ is $-\csc r \cot r$, so

$$\int \csc r \cot r \, dr = -\csc r + C.$$

30. Let

$$g(x) = \int_1^{x^2} \frac{\sin t}{\sqrt{t}} dt.$$

What is the derivative of g?

Answer: Let $h(u) = \int_1^u \frac{\sin t}{\sqrt{t}} dt$ and let $f(x) = x^2$. Then

$$g(x) = h(f(x)),$$

so we can compute $g'(x)$ using the Chain Rule:

$$g'(x) = h'(f(x))f'(x).$$

Now,

$$f'(x) = 2x$$

and, by the Fundamental Theorem of Calculus,

$$h'(u) = \frac{\sin u}{\sqrt{u}}.$$

Hence,

$$g'(x) = h'(f(x))f'(x) = h'(x^2) \cdot 2x = \frac{\sin(x^2)}{\sqrt{x^2}} \cdot 2x = \frac{2x\sin(x^2)}{x} = 2\sin(x^2).$$

31. If $f(x) = \frac{x}{\ln x}$, find $f'(e^3)$.

Answer: Using the quotient rule,

$$f'(x) = \frac{\ln x \cdot \frac{d}{dx}(x) - x \cdot \frac{d}{dx}(\ln x)}{(\ln x)^2} = \frac{\ln x \cdot 1 - x \cdot \frac{1}{x}}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2}.$$

Therefore,

$$f'(e^3) = \frac{\ln(e^3) - 1}{(\ln(e^3))^2} = \frac{3-1}{3^2} = \frac{2}{9}.$$

32. At what value(s) of x (if any) is the function $f(x)$ defined below discontinuous?

$$f(x) = \begin{cases} x^2 + 4x + 5 & \text{if } x < -2 \\ \frac{x}{2} & \text{if } -2 \leq x \leq 2 \\ 1 + \sqrt{x-2} & \text{if } x > 2 \end{cases}$$

For $x < -2$, $f(x) = x^2 + 4x + 5$, which is a polynomial and, hence, continuous. Also, for $-2 < x < 2$, $f(x) = \frac{x}{2}$, which is also a polynomial and hence continuous. Finally, for $x > 2$, $f(x) = 1 + \sqrt{x-2}$, which is continuous since it's the composition of continuous functions (specifically, $f(x) = g \circ h(x)$ where $g(u) = 1 + \sqrt{u}$ and $h(x) = x - 2$). Therefore, the only possible points of discontinuity for f are at $x = -2$ and $x = 2$.

At $x = -2$, $f(x)$ will be continuous if

$$\lim_{x \to -2} f(x) = f(-2) = \frac{-2}{2} = -1.$$

Clearly, $\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} \frac{x}{2} = -1$. On the other hand,

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} \left(x^2 + 4x + 5\right) = (-2)^2 + 4(-2) + 5 = 4 - 8 + 5 = 1.$$

Therefore, since $\lim_{x\to -2^+} f(x) = -1$ and $\lim_{x\to -2^-} f(x) = 1$, we can see that $f(x)$ is *not* continuous at $x = -2$.

Turning to $x = 2$, $f(x)$ will be continuous if

$$\lim_{x\to 2} f(x) = f(2) = \frac{2}{2} = 1.$$

Now $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x}{2} = 1$. On the other hand

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \left(1 + \sqrt{x-2}\right) = 1.$$

Since the limits from the left and the right are both equal to 1, we conclude that

$$\lim_{x\to 2} f(x) = 1,$$

so $f(x)$ *is* continuous at $x = 2$.

Therefore, $f(x)$ is continuous at all real numbers except $x = -2$.

33. Find the equation of the tangent line to the curve

$$x\cos y = 1$$

at the point $\left(2, \frac{\pi}{3}\right)$.

Answer: We will find the slope of the tangent line using implicit differentiation. Differentiating both sides yields

$$1\cdot\cos y + x\cdot(-\sin y\cdot y') = 0$$

or

$$\cos y - xy'\sin y = 0.$$

Therefore,

$$\cos y = xy'\sin y$$

and so we have that

$$y' = \frac{\cos y}{x\sin y}.$$

Therefore, at the point $\left(2, \frac{\pi}{3}\right)$ the slope of the tangent line is

$$y' = \frac{\cos(\pi/3)}{2\cdot\sin(\pi/3)} = \frac{1/2}{2\cdot\frac{\sqrt{3}}{2}} = \frac{1}{2\sqrt{3}}.$$

By the point-slope formula, the equation of the tangent line is

$$y - \frac{\pi}{3} = \frac{1}{2\sqrt{3}}(x-2)$$

or

$$y - \frac{\pi}{3} = \frac{x}{2\sqrt{3}} - \frac{1}{\sqrt{3}}.$$

Therefore, the equation of the tangent line is

$$y = \frac{x}{2\sqrt{3}} - \frac{1}{\sqrt{3}} + \frac{\pi}{3}.$$

34. A particle moves so that its position at time t is given by

$$s(t) = \sqrt{3t^2 + 4}.$$

If $v(t)$ is the velocity of the particle at time t, what is $\lim_{t\to+\infty} v(t)$?

Answer: First, we know that the velocity $v(t)$ is just the derivative of the position function. In other words,

$$v(t) = s'(t) = \frac{1}{2}(3t^2+4)^{-1/2} \cdot 6t = \frac{3t}{\sqrt{3t^2+4}}.$$

Therefore,

$$\lim_{t\to+\infty} v(t) = \lim_{t\to+\infty} \frac{3t}{\sqrt{3t^2+4}}.$$

Now, notice that the leading terms of both numerator and denominator are essentially t (since the t^2 in the denominator is under the square root). Therefore, we should simplify things by multiplying both numerator and denominator by $\frac{1}{t}$:

$$\begin{aligned}
\lim_{t\to+\infty} \frac{3t}{\sqrt{3t^2+4}} &= \lim_{t\to+\infty} \frac{3t}{\sqrt{3t^2+4}} \cdot \frac{\frac{1}{t}}{\frac{1}{t}} \\
&= \lim_{t\to+\infty} \frac{\frac{1}{t}\cdot 3t}{\sqrt{\frac{1}{t^2}\left(3t^2+4\right)}} \\
&= \lim_{t\to+\infty} \frac{3}{\sqrt{3+\frac{4}{t^2}}} \\
&= \frac{3}{\sqrt{3}} \\
&= \sqrt{3}.
\end{aligned}$$

Therefore, we can conclude that $\lim_{t\to+\infty} v(t) = \sqrt{3}$.

35. A technical writer is producing a book which must have 1-inch side margins and 2-inch top and bottom margins. If the area of the page can be at most 50 in^2, what dimensions give the most printed area per page?

Answer: If we call the width of the page x and the height of the page y, then x consists of two 1-inch margins plus the width of the text, so the actual width of the text is $x - 2$. Likewise, y consists of two 2-inch margins plus the height of the text, so the height of the text is $y - 4$.

Now, we know that the area of the page is 50 in^2, meaning that

$$xy = 50.$$

Solving for y, we see that

$$y = \frac{50}{x}.$$

The quantity we want to maximize is the area of the printed text, which, by the discussion above is

$$(x-2)(y-4) = (x-2)\left(\frac{50}{x} - 4\right) = 50 - \frac{100}{x} - 4x + 8 = 58 - \frac{100}{x} - 4x.$$

Hence, we're looking to maximize the function $A(x) = 58 - \frac{100}{x} - 4x$. Of course, it must be the case that $x > 0$, and in fact this is the only constraint (since $x > 0$, $y = \frac{50}{x}$ is automatically bigger than zero).

To find the maximum, let's determine the critical points:

$$A'(x) = \frac{100}{x^2} - 4.$$

Therefore, $A'(x) = 0$ when

$$\frac{100}{x^2} - 4 = 0,$$

meaning that

$$\frac{100}{x^2} = 4,$$

and so

$$100 = 4x^2.$$

Hence $x^2 = 25$ and so $x = \pm 5$. Clearly -5 is not greater than zero, so the only critical point we care about will be $x = 5$.

Since $A''(x) = -\frac{200}{x^3}$, which is negative for all $x > 0$, we see that $x = 5$ is a local maximum. Since it's the only critical point in the interval we're interested, that means it must be the absolute maximum.

Therefore, the printed area will be maximized when the width of the page is $x = 5$ and when the height of the page is $y = \frac{50}{x} = \frac{50}{5} = 10$.

36. Suppose $f(t) = 2t + \cos t$. Evaluate the definite integral

$$\int_{-\pi/2}^{\pi/2} f(t)dt.$$

Answer: Since $t^2 + \sin t$ is an antiderivative of $f(t)$, the Fundamental Theorem of Calculus tells us that

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} (2t + \cos t)dt &= \left[t^2 + \sin t\right]_{-\pi/2}^{\pi/2} \\ &= \left(\left(\frac{\pi}{2}\right)^2 + \sin(\pi/2)\right) - \left(\left(-\frac{\pi}{2}\right)^2 + \sin(-\pi/2)\right) \\ &= \frac{\pi^2}{4} + 1 - \frac{\pi^2}{4} - (-1) \\ &= 2.\end{aligned}$$

37. Evaluate the limit

$$\lim_{x\to\infty} \frac{\ln(3 + 2e^{3x})}{6x}.$$

Answer: Notice that

$$\lim_{x\to\infty} \ln(3 + 2e^{3x}) = \infty$$

and

$$\lim_{x\to\infty} 6x = \infty.$$

Therefore, we can apply L'Hôpital's Rule, so the above limit is equal to

$$\lim_{x\to\infty} \frac{\frac{1}{3+2e^{3x}} \cdot 2e^{3x} \cdot 3}{6} = \lim_{x\to\infty} \frac{\frac{e^{3x}}{3+2e^{3x}}}{1} = \lim_{x\to\infty} \frac{e^{3x}}{3 + 2e^{3x}}.$$

Again, both numerator and denominator are going to ∞, so we can apply L'ôpital's Rule again to get

$$\lim_{x\to\infty} \frac{e^{3x}\cdot 3}{2e^{3x}\cdot 3} = \lim_{x\to\infty} \frac{1}{2} = \frac{1}{2}.$$

Therefore, we conclude that

$$\lim_{x\to\infty} \frac{\ln(3+2e^{3x})}{6x} = \frac{1}{2}.$$

38. As a spherical raindrop falls, it evaporates (i.e. loses volume) at a rate proportional to its surface area. Show that the radius of the raindrop decreases at a constant rate. (*Hint*: the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$, and the surface area is $4\pi r^2$)

    Answer: First, let's record what we know. We know that the volume is given by $V(t) = \frac{4}{3}\pi(r(t))^3$ and that the surface area is given by $A(t) = 4\pi(r(t))^2$. Moreover, since the volume changes at a rate proportional to the surface area, we know that

    $$V'(t) = C \cdot A(t)$$

    for some constant C.

    Now, what we're trying to show is that the radius decreases at a constant rate; in other words, that $r'(t)$ is constant.

    To get at $r'(t)$, let's differentiate $V(t)$:

    $$V'(t) = \frac{4}{3}\pi \cdot 3(r(t))^2 \cdot r'(t) = 4\pi(r(t))^2 r'(t)$$

    by the Chain Rule. On the other hand, we know that $V'(t) = C \cdot A(t)$, so we have that

    $$\begin{aligned} 4\pi(r(t))^2 r'(t) &= C \cdot A(t) \\ 4\pi(r(t))^2 r'(t) &= C \cdot 4\pi(r(t))^2. \end{aligned}$$

    Dividing both sides by $4\pi(r(t))^2$ yields that

    $$r'(t) = C,$$

    so indeed the rate of change of the radius is constant.

39. Suppose $f'(t) = 3e^t - 2\sec t \tan t$ and that $f(0) = 4$. Give a formula for $f(t)$.

    Answer: First, let's take the indefinite integral of $f'(t)$:

    $$\int f'(t)dt = \int \left(3e^t - 2\sec t \tan t\right) dt = 3e^t - 2\sec t + C$$

    for some constant C. In other words,

    $$f(t) = 3e^t - 2\sec t + C$$

    for some C. To determine C, we want to plug in the initial value:

    $$\begin{aligned} f(0) &= 3e^0 - 2\sec(0) + C \\ 4 &= 3 - 2 \cdot 1 + C \\ 4 &= 1 + C, \end{aligned}$$

    so we see that $C = 3$. Therefore, the formula for $f(t)$ is

    $$f(t) = 3e^t - 2\sec t + 3.$$