Fall 2011 Math 2250 Final Exam Solutions

1. Let $f(x) = 2x \ln x$.

   (a) What is the domain of the function f?

   Answer: We're allowed to put any x into $2x$, but we can't just put any x into $\ln x$. In particular, $\ln x$ only makes sense for $x > 0$; therefore, the domain of f is $\{x : x > 0\} = (0, +\infty)$.

   (b) What is $f''(2)$?

   Answer: Using the Product Rule,

$$f'(x) = 2 \cdot \ln x + 2x \cdot \frac{1}{x} = 2\ln x + 2.$$

   Therefore,

$$f''(x) = 2 \cdot \frac{1}{x} = \frac{2}{x},$$

   and so

$$f''(2) = \frac{2}{2} = 1.$$

2. Evaluate the limit

$$\lim_{t\to 1} \frac{(\ln t)^2}{4t^3 - 12t + 8}.$$

   Answer: Notice that $\lim_{t\to 1}(\ln t)^2 = 0$ and likewise

$$\lim_{t\to 1}\left(4t^3 - 12t + 8\right) = 0.$$

   Therefore, we can apply L'Hôpital's Rule to see that the above limit is equal to

$$\lim_{t\to 1} \frac{2\ln t \cdot \frac{1}{t}}{12t^2 - 12} = \lim_{t\to 1} \frac{2\ln t}{12t^3 - 12t}.$$

   Now, the numerator and denominator are both still going to zero, so we apply L'Hôpital's Rule again to see that the above limit is equal to

$$\lim_{t\to 1} \frac{2 \cdot \frac{1}{t}}{36t^2 - 12} = \lim_{t\to 1} \frac{2}{36t^3 - 12t} = \frac{2}{36 - 12} = \frac{2}{24} = \frac{1}{12}.$$

   Therefore, we can conclude that

$$\lim_{t\to 1} \frac{(\ln t)^2}{4t^3 - 12t + 8} = \frac{1}{12}.$$

3. Consider the function

$$f(x) = \begin{cases} \cos\left(\frac{\pi x}{3}\right) & \text{if } |x| \leq 1 \\ \frac{x}{2} & \text{if } |x| > 1. \end{cases}$$

   For what values of x is $f(x)$ discontinuous?

   Answer: Since the cosine function and the linear function $g(x) = x$ are both continuous everywhere, the function f is certainly continuous away from $x = 1$ and $x = -1$. To check whether f is continuous at $x = 1$, we just need to see whether $\lim_{x\to 1} f(x) = f(1)$. Since $f(1) = \cos(\pi/3) = 1/2$, this amounts to checking whether $\lim_{x\to 1} f(x) = 1/2$. In turn, this will be true if and only if the two one-sided limits $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$ are both equal to $1/2$, which is easy to check:

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x}{2} = \frac{1}{2}$$
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \cos\left(\frac{\pi x}{3}\right) = \frac{1}{2}.$$

Therefore we can conclude that $\lim_{x\to 1} f(x) = 1/2$ and so f is continuous at $x = 1$.

On the other hand, to check whether f is continuous at $x = -1$, we need to see whether $\lim_{x\to -1} f(x) = f(-1)$. Now $f(-1) = \cos(-\pi/3) = 1/2$, but the one-sided limit

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \frac{x}{2} = -\frac{1}{2},$$

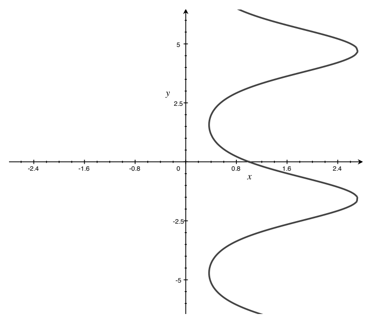
so f is *not* continuous at $x = -1$.

Therefore, the function f is discontinuous only at $x = -1$.

4. The curve shown below is described by the equation

$$xe^{\sin y} = 1.$$

What is the slope of the tangent line to the curve at the point $(1, 0)$?



Answer: We can determine the slope of the curve by computing y' using implicit differentiation. Differentiating both sides yields

$$1 \cdot e^{\sin y} + xe^{\sin y} \cdot \cos y \cdot y' = 0.$$

Therefore,

$$xy' \cos y e^{\sin y} = -e^{\sin y},$$

so

$$y' = \frac{-e^{\sin y}}{x \cos y e^{\sin y}} = \frac{-1}{x \cos y}.$$

This holds at any point (x, y) on the curve; at the point $(1, 0)$ we get that

$$y' = \frac{-1}{1 \cos(0)} = \frac{-1}{1} = -1,$$

so the tangent line to the curve at the point $(1, 0)$ has slope -1.

5. A bacterial culture is being grown in a Petri dish, and the area is increasing at a rate of 1 cm^2 per day. At what rate is the circumference of the culture increasing when the radius is 2 cm?

Answer: First, let's list the things we know. If $A(t)$ is the area of the culture after t days, then

$$A(t) = \pi r(t)^2.$$

Likewise, if $C(t)$ is the circumference of the culture after t days, then

$$C(t) = 2\pi r(t).$$

Also, we're told that $A'(t) = 1$ for all t.

Now, if t_0 is the time when the area of the culture is 4π cm^2, then we're asked to determine $C'(t_0)$.

Notice that

$$C'(t) = 2\pi r'(t),$$

and so in particular $C'(t_0) = 2\pi r'(t_0)$. Hence, if we can determine $r'(t_0)$, we can just plug it into the above equation and get our answer.

To get a handle on $r'(t_0)$, let's differentiate the expression for $A(t)$:

$$A'(t) = 2\pi r(t) r'(t).$$

Plugging in $t = t_0$, we see that

$$\begin{aligned} A'(t_0) &= 2\pi r(t_0) r'(t_0) \\ 1 &= 2\pi r(t_0) r'(t_0) \end{aligned}$$

since we know $A'(t_0) = 1$. Therefore,

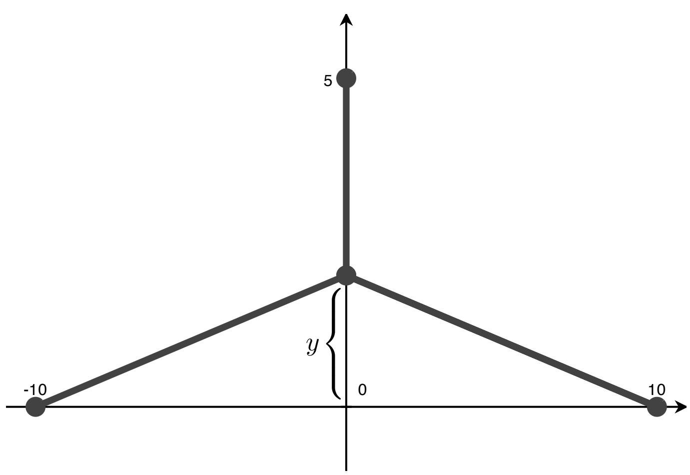
$$r'(t_0) = \frac{1}{2\pi r(t_0)} = \frac{1}{2\pi \cdot 2} = \frac{1}{4\pi}.$$

Now we just plug this into the expression for $C'(t_0)$:

$$C'(t_0) = 2\pi r'(t_0) = 2\pi \cdot \frac{1}{4\pi} = \frac{1}{2},$$

so the circumference is increasing at the rate of $\frac{1}{2}$ cm per day when the radius is equal to 2 cm.

6. Two factories are located at the coordinates $(10, 0)$ and $(-10, 0)$ and the local power station is located at $(0, 5)$, as shown in the figure. Find the distance y that minimizes the total length of power lines (the thick lines in the figure) from the power station to the factories. (*Note: the approximations* $\sqrt{3} \approx 1.73$ *and* $\sqrt{125} \approx 11.2$ *may come in handy.*)



Answer: Notice that the vertical stretch of power line has length $5 - y$. On the other hand, each of the two diagonal stretches of power line has length

$$\sqrt{10^2 + y^2} = \sqrt{100 + y^2}$$

by the Pythagorean Theorem. Therefore, the total length of power line is

$$P(y) = 5 - y + 2\sqrt{100 + y^2}.$$

We want to find the absolute minimum of this function subject to the following constraints. First, $0 \le y$ since a length cannot be negative. On the other hand, $y \le 5$ since the diagonal power lines can't come together above the power station. Therefore, we want to minimize $P(y) = 5 - y + 2\sqrt{100 + y^2}$ on the interval $[0, 5]$.

First, differentiate to find critical points:

$$P'(y) = -1 + 2\frac{1}{2\sqrt{100 + y^2}} \cdot 2y = -1 + \frac{2y}{\sqrt{100 + y^2}}.$$

Therefore, we'll have a critical point when

$$0 = -1 + \frac{2y}{\sqrt{100 + y^2}},$$

which is to say when

$$1 = \frac{2y}{\sqrt{100 + y^2}}$$

or, equivalently,

$$\sqrt{100 + y^2} = 2y.$$

We want to solve for y, so square both sides

$$100 + y^2 = 4y^2,$$

so we see that

$$3y^2 = 100,$$

and hence $y = \pm\sqrt{\frac{100}{3}} = \pm\frac{10}{\sqrt{3}}$.

We can ignore $-\frac{10}{\sqrt{3}}$ since it lies outside the interval $[0, 5]$. In fact, since $\sqrt{3} < 2$, we know that

$$\frac{10}{\sqrt{3}} > \frac{10}{2} = 5,$$

so *both* critical points lie outside the interval.

Hence, to find the absolute minimum, we only need to check the endpoints:

$$P(0) = 5 - 0 + 2\sqrt{100 + 0^2} = 5 + 20 = 25$$
$$P(5) = 5 - 5 + 2\sqrt{100 + 5^2} = 0 + 2\sqrt{125} = 2\sqrt{125}.$$

Therefore, the length of power lines is minimized when $y = 5$, meaning the diagonal lines actually meet at the power station and there is no vertical segment.

7. Use an appropriate linearization to approximate $\sqrt[4]{0.98}$.

Answer: Let $f(x) = \sqrt[4]{x} = x^{1/4}$. Then the linearization of $f(x)$ at $x = 1$ is

$$L(x) = f(1) + f'(1)(x - 1).$$

Now, $f(1) = \sqrt[4]{1} = 1$. Also,

$$f'(x) = \frac{1}{4}x^{-3/4},$$

so $f'(1) = \frac{1}{4} \cdot 1^{-3/4} = \frac{1}{4}$, so the above linearization is

$$L(x) = 1 + \frac{1}{4}(x - 1).$$

We want to approximate $\sqrt[4]{0.98}$, which is just $f(0.98)$. Therefore,

$$\sqrt[4]{0.98} = f(0.98) \approx L(0.98) = 1 + \frac{1}{4}(0.98 - 1) = 1 + \frac{-0.02}{4} = 1 - 0.005 = 0.995.$$

8. Evaluate

$$\int_0^{\sqrt{3}} \frac{x}{\sqrt{x^2+1}}\, dx.$$

Answer: Let $u = x^2 + 1$. Then $du = 2x\, dx$. Since

$$\int_0^{\sqrt{3}} \frac{x}{\sqrt{x^2+1}}\, dx = \frac{1}{2}\int_0^{\sqrt{3}} \frac{1}{\sqrt{x^2+1}}\, 2x\, dx$$

and since $u(\sqrt{3}) = \sqrt{3}^2 + 1 = 4$ and $u(0) = 0^2 + 1 = 1$, the given integral is equal to

$$\frac{1}{2}\int_1^4 \frac{1}{\sqrt{u}}\, du = \frac{1}{2}\int_1^4 u^{-1/2} du = \frac{1}{2}\left[2u^{1/2}\right]_1^4 = \sqrt{4} - \sqrt{1} = 2 - 1 = 1.$$

9. Suppose the velocity of a particle is modeled by the function $v(t) = 2t^2 - 6t + 4$, where t is measured in seconds and velocity in meters per second.

(a) If the position of the particle at time $t = 0$ is 4 meters to the right of the origin, what is the function $s(t)$ which describes the position of the particle after t seconds?
Answer: Since $v(t) = s'(t)$, we want to integrate $v(t)$ to determine $s(t)$:

$$\int v(t)\, dt = \int (2t^2 - 6t + 4) dt = \frac{2}{3}t^3 - 3t^2 + 4t + C.$$

Therefore, $s(t) = \frac{2}{3}t^3 - 3t^2 + 4t + C$ for some constant C. To determine C, evaluate at $t = 0$:

$$\begin{aligned} s(0) &= \frac{2}{3} \cdot 0^3 - 3 \cdot 0^2 + 4 \cdot 0 + C \\ 4 &= C. \end{aligned}$$

Hence, the position of the particle is given by the function

$$s(t) = \frac{2}{3}t^3 - 3t^2 + 4t + 4.$$

(b) After how many seconds does the particle change direction?
Answer: The particle changes direction when the velocity of the particle changes sign. Now, since

$$v(t) = 2t^2 - 6t + 4 = (2t - 2)(t - 2),$$

we see that the velocity changes sign at $t = 1$ and $t = 2$, so the particle changes direction after 1 second and again after 2 seconds.

10. Sketch the graph of the function $f(x) = \frac{x}{e^x}$. You should be sure to label any absolute or relative maxima or minima and any inflection points, and your sketch should be good enough that it's clear where the graph is concave up and where it is concave down. Any asymptotes should also be clear in your sketch. (*Hint: it's a good idea to factor and simplify as much as possible at each stage.*)

Answer: Notice, first of all, that

$$\lim_{x \to -\infty} \frac{x}{e^x} = -\infty$$

since the numerator is going to $-\infty$ and the denominator is positive but going to zero.

On the other hand, by L'Hôpital's Rule,

$$\lim_{x \to +\infty} \frac{x}{e^x} = \lim_{x \to +\infty} \frac{1}{e^x} = 0,$$

so $f(x)$ has a horizontal asymptote at $y = 0$ as $x \to +\infty$.

To find maxima and minima, let's differentiate and find critical points:

$$f'(x) = \frac{e^x \cdot 1 - xe^x}{e^{2x}} = \frac{e^x(1-x)}{e^{2x}} = \frac{1-x}{e^x}.$$

Therefore, since the denominator is always positive, we have a critical point only when $1 - x = 0$, meaning when $x = 1$.

To determine whether this is a max or a min, let's use the second derivative test.

$$f''(x) = \frac{e^x(-1) - (1-x)e^x}{e^{2x}} = \frac{e^x(-1-1+x)}{e^{2x}} = \frac{x-2}{e^x}.$$

Therefore, $f''(x) < 0$ when $x < 2$ and $f''(x) > 0$ when $x > 2$. In particular, $f''(1) < 0$, so $f(x)$ has a local maximum at $x = 1$ and the function is concave down for $x < 2$ and concave up for $x > 2$, meaning there is an inflection point at $x = 2$.

Putting this all together, I can graph the function, which has an absolute max at $\left(1, \frac{1}{e}\right)$ and an inflection point at $\left(2, \frac{2}{e^2}\right)$, which are the two red dots.