Math 2250 Exam #3 Solutions

1. The function $g(x) = x^3 - 2x^2 - 4x + 9$ has no absolute minimum, but what is the local minimum value of $g(x)$?

   Answer: To find the local minimum, first we want to determine the critical points of g, so we take the derivative:

$$g'(x) = 3x^2 - 4x - 4 = (3x+2)(x-2)$$

   Therefore, $g'(x) = 0$ when $(3x+2)(x-2) = 0$, meaning when $x = -2/3$ or $x = 2$. To determine which is the local minimum, we'll use the Second Derivative Test.

$$g''(x) = 6x - 4,$$

   so

$$g''(-2/3) = 6(-2/3) - 4 = -8$$
$$g''(2) = 6(2) - 4 = 8.$$

   Therefore, by the Second Derivative Test, since $x = 2$ is a critical point and $g''(2) > 0$, we know g has a local minimum at $x = 2$. Therefore, the local minimum value of g is

$$g(2) = 2^3 - 2(2)^2 - 4(2) + 9 = 8 - 8 - 8 + 9 = 1.$$

2. Evaluate the limit

$$\lim_{x \to 0^+} \sin(x) \tan(x + \pi/2).$$

   Answer: Notice that $\lim_{x\to 0^+} \sin(x) = \sin(0) = 0$ since the sine function is continuous, while

$$\lim_{x\to 0^+} \tan(x+\pi/2) = \lim_{x\to 0^+} \frac{\sin(x+\pi/2)}{\cos(x+\pi/2)} = -\infty$$

   since the numerator goes to 1 and the denominator is negative but approaching zero.

   Therefore, the limit we're computing is of the form $0 \cdot \infty$, so before we can use L'Hôpital's Rule, we need to re-write it in the following form:

$$\lim_{x\to 0^+} \sin(x)\tan(x+\pi/2) = \lim_{x\to 0^+} \frac{\sin(x)}{\frac{1}{\tan(x+\pi/2)}} = \lim_{x\to 0^+} \frac{\sin(x)}{\cot(x+\pi/2)}.$$

   Now both numerator and denominator are going to zero, so, by L'Hôpital's rule, the above limit is equal to

$$\lim_{x\to 0^+} \frac{\frac{d}{dx}(\sin(x))}{\frac{d}{dx}(\cot(x+\pi/2))} = \lim_{x\to 0^+} \frac{\cos(x)}{-\csc^2(x+\pi/2)}.$$

   Using the fact that $\csc(x+\pi/2) = \frac{1}{\sin(x+\pi/2)}$, the above is equal to

$$\lim_{x\to 0^+} \frac{\cos(x)}{\frac{1}{-\sin^2(x+\pi/2)}} = \lim_{x\to 0^+} \cos(x)\left(-\sin^2(x+\pi/2)\right) = -\cos(0)\sin^2(\pi/2) = -1$$

   since cosine and sine are both continuous everywhere.

   Putting all this together, then, we conclude that

$$\lim_{x\to 0^+} \sin(x)\tan(x+\pi/2) = -1.$$

(Note: you could also evaluate this limit by noticing that $\tan(x+\pi/2) = \frac{\sin(x+\pi/2)}{\cos(x+\pi/2)}$, and so the given limit is equal to $\lim_{x\to 0^+} \frac{\sin(x)\sin(x+\pi/2)}{\cos(x+\pi/2)}$, which can be evaluated using L'Hôpital's Rule. Even sneakier would be to use the trig identities $\sin(x+\pi/2) = -\cos(x)$ and $\cos(x+\pi/2) = \sin(x)$, so

$$\lim_{x\to 0^+} \frac{\sin(x)\sin(x+\pi/2)}{\cos(x+\pi/2)} = \lim_{x\to 0^+} \frac{-\sin(x)\cos(x)}{\sin(x)} = \lim_{x\to 0^+} -\cos(x),$$

which is clearly equal to -1.)

3. Let $f(x) = e^{-2x^2}$. Sketch a graph of $f(x)$. Be sure to label any absolute maxima or minima and any inflection points. Make sure your sketch is good enough that it's clear where the function is concave up and where it is concave down.

Answer: First, notice that

$$\lim_{x\to+\infty} e^{-2x^2} = \lim_{x\to+\infty} \frac{1}{e^{2x^2}} = 0,$$

and likewise

$$\lim_{x\to-\infty} e^{-2x^2} = \lim_{x\to-\infty} \frac{1}{e^{2x^2}} = 0,$$

so $f(x)$ has horizontal asymptotes at $y = 0$ in both directions. In particular, this means the largest relative maximum will actually be the absolute maximum and the smallest relative minimum will actually be the absolute minimum.

Now, we take the derivative to find critical points:

$$g'(x) = e^{-2x^2} \cdot (-4x) = -4xe^{-2x^2}.$$

Since e^{-2x^2} is always positive, $g'(x) = 0$ only when $x = 0$, so this is the only critical point. To see whether this is a maximum or minimum, we'll use the Second Derivative Test:

$$g''(x) = -4e^{-2x^2} - 4xe^{-2x^2}(-4x) = -4e^{-2x^2} + 16x^2e^{-2x^2} = -4e^{-2x^2}(1-4x^2).$$

Therefore,

$$g''(0) = -4e^{-2\cdot 0^2}(1-4(0)^2) = -4(1) = -4,$$

so, by the second derivative test, g has a relative maximum at $x = 0$. In fact, since this is the only relative maximum, the earlier discussion implies that g has an absolute maximum at $x = 0$, and the absolute maximum value is

$$g(0) = e^{-2\cdot 0^2} = 1.$$

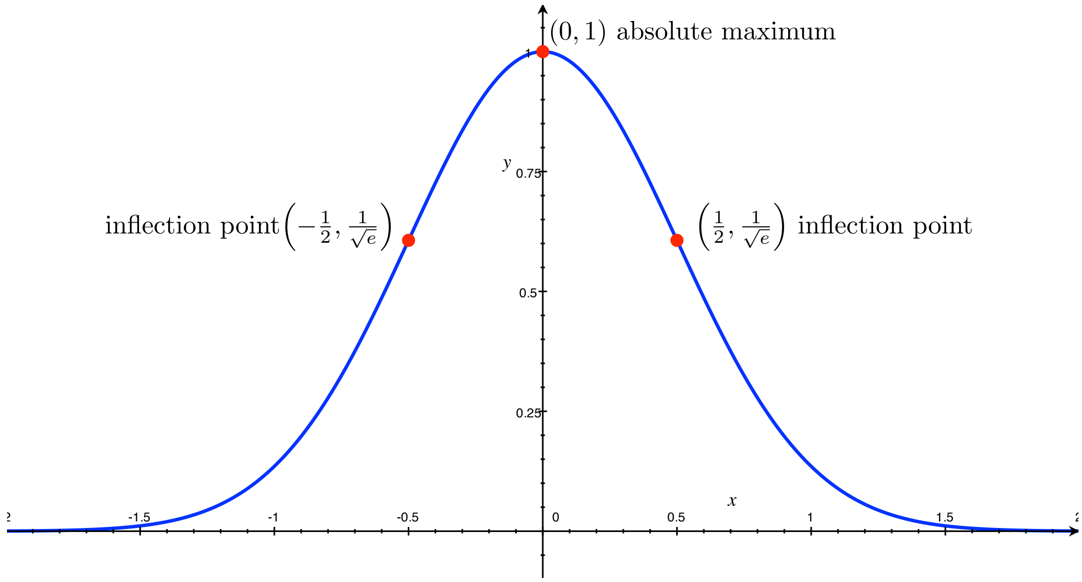
Since there are no other critical points and g is differentiable everywhere, there is no absolute minimum.

To find inflection points of g, notice that $g''(x) = 0$ when

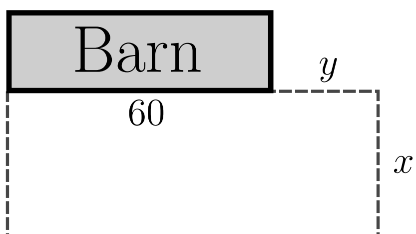
$$-4e^{-2x^2}(1-4x^2) = 0.$$

Since $-4e^{-2x^2}$ is never zero, this only happens when $1 - 4x^2 = 0$, which is to say when $4x^2 = 1$, meaning $x^2 = 1/4$ and $x = \pm 1/2$. Clearly, $g''(x)$ changes from positive to negative at $x = -1/2$ and from negative to positive at $x = 1/2$, so these are both inflection points and $g(x)$ is concave up on $(-\infty, -1/2)$ and $(1/2, +\infty)$ and concave down on $(-1/2, 1/2)$.

Putting all this together, then, the graph of $g(x)$ looks like the figure at the top of the following page.



4. A farmer plans to construct a pen next to his barn, using all of the barn as part of one side of the pen. If the barn is 60 feet long and the farmer has 300 feet of fencing material, find the dimensions of the pen with the largest area that the farmer can build. Note: There is no fence along the barn wall.



Answer: First, notice that the length of fencing is equal to

$$y + x + (y + 60) + x = 2y + 2x + 60.$$

Since the farmer has 300 feet of fence, we know that

$$2y + 2x + 60 = 300,$$

meaning that $2y + 2x = 240$, so $2y = 240 - 2x$ and hence $y = 120 - x$.

Now, the area of the pen will be $x(60+y)$. Since $y = 120 - x$, we see that the area as a function of x is

$$A(x) = x(60 + (120 - x)) = x(180 - x) = 180x - x^2.$$

Now, we want to maximize the function $A(x)$ subject to the constraint that $x \geq 0$ and $y \geq 0$. This latter implies that $120 - x \geq 0$, so $x \leq 120$. Therefore, we're maximizing $A(x)$ on the closed interval $[0, 120]$.

First, find critical points:

$$A'(x) = 180 - 2x,$$

so $A'(x) = 0$ when $180 - 2x = 0$, meaning $180 = 2x$ or, equivalently, $x = 90$. Hence, the only critical point is at $x = 90$.

Now, just evaluate $A(x)$ at the critical point and the endpoints:

$$\begin{aligned} A(0) &= 0(180 - 0) = 0 \\ A(90) &= 90 \cdot (180 - 90) = 90^2 = 8100 \\ A(120) &= 120 \cdot (180 - 120) = 120 \cdot 60 = 7200. \end{aligned}$$

Therefore, the maximum area of 8100 square feet is achieved when $x = 90$ and $y = 120 - 90 = 30$. Notice that this means the pen is square, which perhaps isn't so surprising.

5. A certain bacterial culture starts with a mass of 10 milligrams. If the rate of increase of the mass (in mg/hour) is described by the function $\frac{1}{t+1} + \frac{1}{\sqrt{t+4}}$, what is the mass of the culture after t hours? (*Note: finding a common denominator is probably going to lead you astray.*)

   Answer: Let $M(t)$ be the mass in milligrams of the bacterial culture after t hours. Then we're told that $M(0) = 10$ and that

$$M'(t) = \frac{1}{t+1} + \frac{1}{\sqrt{t+4}}.$$

   To solve this initial value problem, we need to find an antiderivative for the above function. Notice, first of all, that

$$\frac{d}{dt}(\ln(t+1)) = \frac{1}{t+1},$$

   so $\ln(t+1)$ is an antiderivative for $\frac{1}{t+1}$.

   Also,

$$\frac{d}{dt}\left(\sqrt{t+4}\right) = \frac{d}{dt}\left((t+4)^{1/2}\right) = \frac{1}{2}(t+4)^{-1/2} = \frac{1}{2} \cdot \frac{1}{\sqrt{t+4}}.$$

   Therefore, $2\sqrt{t+4}$ is an antiderivative for $\frac{1}{\sqrt{t+4}}$.

   This all means that $\ln(t+1) + 2\sqrt{t+4}$ is an antiderivative for $M'(t) = \frac{1}{t+1} + \frac{1}{\sqrt{t+4}}$, and so

$$M(t) = \ln(t+1) + 2\sqrt{t+4} + C$$

   for some constant C. To determine C, we'll use the fact that $M(0) = 10$:

$$\begin{aligned} M(0) &= \ln(0+1) + 2\sqrt{0+4} + C \\ 10 &= 0 + 4 + C, \end{aligned}$$

   so we see that $C = 6$. Hence, the mass of the culture after t hours is given by

$$M(t) = \ln(t+1) + 2\sqrt{t+4} + 6.$$