Fall 2011 Math 2250 Exam #2 Solutions

1. Find the derivatives of the following functions.

   (a) $f(x) = \ln(\arctan(x))$
   Answer: Using the Chain Rule,

$$f'(x) = \frac{1}{\arctan(x)} \frac{d}{dx}(\arctan(x))$$

   Remembering that the derivative of $\arctan(x)$ is $\frac{1}{1+x^2}$, the above becomes

$$f'(x) = \frac{1}{\arctan(x)} \frac{1}{1+x^2}.$$

   (b) $g(x) = \frac{\sin(x)}{\cos^2(x)}$
   Answer: The hard way to solve this is using the Quotient Rule:

$$\begin{aligned} g'(x) &= \frac{\cos^2(x) \cdot \frac{d}{dx}(\sin(x)) - \sin(x) \cdot \frac{d}{dx}(\cos^2(x))}{(\cos^2(x))^2} \\ &= \frac{\cos^2(x)(\cos(x)) - \sin(x)(-2\cos(x) \cdot (-\sin(x)))}{\cos^4(x)} \\ &= \frac{\cos^3(x) + 2\sin^2(x)\cos(x)}{\cos^4(x)} \\ &= \frac{1}{\cos(x)} + 2\frac{\sin^2(x)}{\cos^3(x)}. \end{aligned}$$

   The easy way is to notice that

$$g(x) = \frac{\sin(x)}{\cos^2(x)} = \frac{1}{\cos(x)} \frac{\sin(x)}{\cos(x)} = \sec(x)\tan(x).$$

   Therefore, by the Product Rule,

$$g'(x) = (\sec(x)\tan(x))\tan(x) + \sec(x)(\sec^2(x)) = \sec(x)(\tan^2(x) + \sec^2(x)).$$

   To see that the two answers we got are the same, remember that $\tan^2(x) + 1 = \sec^2(x)$, so the second expression for $g'(x)$ becomes

$$\sec(x)(\tan^2(x) + \tan^2(x) + 1) = \sec(x)(1 + 2\tan^2(x)) = \frac{1}{\cos(x)} + 2\frac{\sin^2(x)}{\cos^3(x)}.$$

2. Let $f(x) = 3^{x\ln(x)}$. What is $f'(0)$?
   Answer: Unless you remember the derivative of 3^x and use the Chain Rule, the best way to solve this is to use logarithmic differentiation. Taking the natural log of both sides, we have

$$\ln(f(x)) = \ln(3^{x\ln(x)}) = x\ln(x)\ln(3).$$

   Differentiating both sides yields

$$\begin{aligned} \frac{1}{f(x)} \cdot f'(x) &= \ln(3)\left(1 \cdot \ln(x) + x \cdot \frac{1}{x}\right) \\ \frac{f'(x)}{f(x)} &= \ln(3)(\ln(x) + 1). \end{aligned}$$

Therefore,

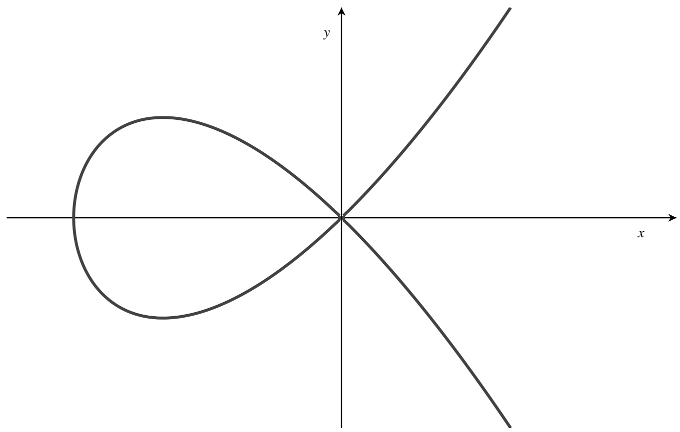
$$f'(x) = f(x)\ln(3)(\ln(x)+1)$$

and, substituting back in for $f(x)$ gives us the derivative of f:

$$f'(x) = 3^{x\ln(x)}\ln(3)(\ln(x)+1).$$

Now, when we try to evaluate at $x = 0$, we run into a problem: $\ln(0)$ is undefined! Therefore, $f'(0)$ is undefined as well (in fact, we could have concluded this right at the beginning, since 0 is not in the domain of f).

3. The curve determined by the equation $y^2 = x^2(x+1)$, pictured below, is called the Tschirnhausen cubic. At what points does this curve have a horizontal tangent line?



Answer: Differentiating implicitly, we have

$$2y \cdot y' = 2x(x+1) + x^2(1).$$

Thus,

$$y' = \frac{2x^2 + 2x + x^2}{2y} = \frac{3x^2 + 2x}{2y}.$$

The curve will have a horizontal tangent line only when the above is equal to zero. Clearly, the fraction can be zero only when the numerator is equal to zero:

$$0 = 3x^2 + 2x = x(3x+2).$$

Hence, the numerator is equal to zero when $x = 0$ or $x = -2/3$. *However* (and this is the slightly tricky bit), when x is equal to zero, so is y: in this case, both numerator *and* denominator are zero, so the expression isn't necessarily zero. In fact, it's clear from the graph above that the tangent line is *not* horizontal at the origin, so the graph has a horizontal tangent line only when $x = -2/3$. Using the original equation of the curve to solve for the y-coordinates corresponding to $x = -2/3$, we see that

$$y^2 = (-2/3)^2(-2/3+1) = \frac{4}{9} \cdot \frac{1}{3} = \frac{4}{27}.$$

Therefore, $y = \pm\sqrt{\frac{4}{27}} = \pm\frac{2}{3\sqrt{3}}$.

We conclude, then, that the Tschirnhausen cubic has a horizontal tangent line at the points

$$\left(-\frac{2}{3}, \frac{2}{3\sqrt{3}}\right) \text{ and } \left(-\frac{2}{3}, -\frac{2}{3\sqrt{3}}\right).$$

4. The volume of a cube increases at a rate of 1 cm^3 per minute. How fast is the surface area of the cube increasing when the length of an edge is 3 cm?

   Answer: First, let's collect the facts that we know. Letting $\ell(t)$ be the length of an edge at time t, $V(t)$ be the volume, $A(t)$ be the surface area, and t_0 be the time when the edge length is 3, we have

$$\begin{aligned} V(t) &= \ell(t)^3 \\ A(t) &= 6\ell(t)^2 \text{ (remember, a cube has six faces)} \\ V'(t) &= 1 \\ \ell(t_0) &= 3 \end{aligned}$$

   Our goal is to determine $A'(t_0)$. Notice that, using the above and the Chain Rule,

$$A'(t) = 6 \cdot 2\ell(t)\ell'(t) = 12\ell(t)\ell'(t).$$

   Therefore, at the time of interest,

$$A'(t_0) = 12\ell(t_0)\ell'(t_0) = 12(3)\ell'(t_0) = 36\ell'(t_0).$$

   If we can determine $\ell'(t_0)$ we'll be able to substitute it into the above expression and determine $A'(t_0)$. We will be able to solve for $\ell'(t_0)$ by differentiating $V(t)$ and evaluating at $t = t_0$. First, we differentiate:

$$V'(t) = 3\ell(t)^2\ell'(t).$$

   Since $V'(t) = 1$ for all t, we know

$$1 = 3\ell(t)^2\ell'(t),$$

   so

$$\ell'(t) = \frac{1}{3\ell(t)^2}.$$

   Now we can just evaluate at $t = t_0$:

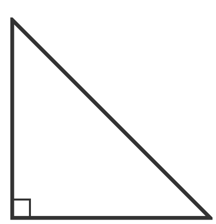
$$\ell'(t_0) = \frac{1}{3(3)^2} = \frac{1}{27}.$$

   Finally, then the expression for $A'(t_0)$ determined above yields

$$A'(t_0) = 36\ell'(t_0) = 36\frac{1}{27} = \frac{4}{3}.$$

   Therefore, at the moment when the edges of the cube have length 3 cm, the surface area is growing at a rate of $\frac{4}{3}$ cm^2/min.

5. Suppose that you are interested in computing the area of the right triangle pictured below and you determine that the right triangle is isosceles (this is easy to do using a compass). If you are able to measure the length of one of the short sides with a maximum error of 1%, how accurately can you compute the area of this right triangle?



Answer: Recall, first, that the area of a triangle is

$$A = \frac{1}{2}bh.$$

In the case of a right triangle like this one, b and h are just the lengths of the two short sides. Moreover, when the right triangle is isosceles, the two shorter sides are the same length; i.e. $b = h$. Therefore

$$A = \frac{1}{2}h \cdot h = \frac{1}{2}h^2.$$

Now, we intend to approximate the error in measuring the area using differentials. First, note that the differential of the area is

$$dA = \frac{1}{2} \cdot 2hdh = hdh.$$

Also, we can measure the length of one of the shorter sides (which is to say, h) with a maximum error of 1%, which means that

$$\frac{dh}{h} \leq 0.01.$$

The maximum relative error in measuring the area, then, is approximately

$$\frac{dA}{A} = \frac{hdh}{\frac{1}{2}h^2} = 2\frac{dh}{h} \leq 2(0.01) = 0.02.$$

Therefore, the approximate maximum percentage error in measuring the area of the triangle is 2%.