Fall 2011 Math 2250 Exam #1 Solutions

1. What is the average rate of change of the function $v(t) = \csc(t)$ over the interval $[\pi/6, \pi/2]$?

   Answer: The average rate of change of $v(t)$ over $[\pi/6, \pi/2]$ is simply

$$\begin{aligned}
\frac{v(\pi/2) - v(\pi/6)}{\pi/2 - \pi/6} &= \frac{\csc(\pi/2) - \csc(\pi/6)}{\pi/3} \\
&= \frac{\frac{1}{\sin(\pi/2)} - \frac{1}{\sin(\pi/6)}}{\pi/3} \\
&= \frac{\frac{1}{1} - \frac{1}{1/2}}{\pi/3} \\
&= \frac{1-2}{\pi/3} \\
&= \frac{-1}{\pi/3} \\
&= -\frac{3}{\pi}.
\end{aligned}$$

2. Determine either of the horizontal asymptotes to the curve

$$y = \frac{3x - \frac{5}{2}}{\sqrt{4x^2 + 6x - 9}}.$$

   Answer: Let $f(x) = \frac{3x-\frac{5}{2}}{\sqrt{4x^2+6x-9}}$. The curve has a horizontal asymptote $y = L$ if $\lim_{x\to+\infty} f(x) = L$ *or* $\lim_{x\to-\infty} f(x) = L$. So that we don't have to worry about minus signs, let's evaluate the limit as x goes to $+\infty$. To do so, we will multiply by $\frac{1/x}{1/x}$ (since the highest power of x is x^2, but it's under a square root so it only counts as degree 1):

$$\begin{aligned}
\lim_{x\to+\infty} \frac{3x - \frac{5}{2}}{\sqrt{4x^2 + 6x - 9}} &= \lim_{x\to+\infty} \frac{3x - \frac{5}{2}}{\sqrt{4x^2 + 6x - 9}} \frac{1/x}{1/x} \\
&= \lim_{x\to+\infty} \frac{\frac{1}{x}\left(3x - \frac{5}{2}\right)}{\sqrt{\frac{1}{x^2}\left(4x^2 + 6x - 9\right)}} \\
&= \lim_{x\to+\infty} \frac{3 - \frac{5}{2x}}{\sqrt{4 + \frac{6}{x} - \frac{9}{x^2}}} \\
&= \frac{3}{\sqrt{4}} \\
&= \frac{3}{2}.
\end{aligned}$$

   Therefore, the curve $y = \frac{3x-\frac{5}{2}}{\sqrt{4x^2+6x-9}}$ has $y = \frac{3}{2}$ as a horizontal asymptote.

   (The other horizontal asymptote, which you can determine by evaluating the limit as $x \to -\infty$, is $y = -\frac{3}{2}$).

3. Let

$$f(x) = 2\frac{x-1}{\sqrt{x^2 - 1}}.$$

   For each of the following, either evaluate the limit or explain why it doesn't exist.

(a)

$$\lim_{x\to 1^+} f(x)$$

Answer: Notice, first of all, that both the numerator and the denominator are going to zero, so we need to get a bit clever. Since $x^2-1=(x-1)(x+1)$, we can write

$$\sqrt{x^2-1}=\sqrt{(x-1)(x+1)}=\sqrt{x-1}\sqrt{x+1}.$$

Therefore,

$$\lim_{x\to 1^+} 2\frac{x-1}{\sqrt{x^2-1}}=\lim_{x\to 1^+} 2\frac{x-1}{\sqrt{x-1}\sqrt{x+1}}=\lim_{x\to 1^+} 2\frac{\sqrt{x-1}}{\sqrt{x+1}}=0,$$

since the numerator is going to zero while the denominator goes to $\sqrt{2}$.

(b)

$$\lim_{x\to 1^-} f(x)$$

Answer: Notice that $\sqrt{x^2-1}$ is undefined for all x between -1 and 1, so $f(x)$ is undefined on this whole interval. Therefore, $f(x)$ is undefined for x approaching 1 from the left, so the limit is also undefined.

(c)

$$\lim_{x\to -1^-} f(x)$$

Answer: We have that

$$\lim_{x\to -1^-} 2\frac{x-1}{\sqrt{x^2-1}}=-\infty$$

since the numerator of the fraction is going to -2 (and in particular is negative for all x to the left of -1) and the denominator is going to 0 and is always positive.

4. Find the equation of the tangent line to the curve $y=\sqrt{x}$ at the point $(4,2)$.

Answer: We know that the slope of a secant line through $(4,2)$ and some nearby point $(4+h,\sqrt{4+h})$ is given by

$$\frac{\sqrt{4+h}-\sqrt{4}}{h}.$$

Letting h go to zero will yield the slope of the tangent line. More formally, the slope of the tangent line is

$$\begin{aligned}\lim_{h\to 0}\frac{\sqrt{4+h}-\sqrt{4}}{h}&=\lim_{h\to 0}\frac{\sqrt{4+h}-\sqrt{4}}{h}\cdot\frac{\sqrt{4+h}+\sqrt{4}}{\sqrt{4+h}+\sqrt{4}}\\&=\lim_{h\to 0}\frac{(4+h)-4}{h(\sqrt{4+h}+\sqrt{4})}\\&=\lim_{h\to 0}\frac{h}{h(\sqrt{4+h}+2)}\\&=\lim_{h\to 0}\frac{1}{\sqrt{4+h}+2}\\&=\frac{1}{4}.\end{aligned}$$

The key step here (in the first line) is to rationalize the numerator.

Now, we can use the point-slope formula to determine the equation of the tangent line:

$$\begin{aligned} y - 2 &= \frac{1}{4}(x-4) \\ y - 2 &= \frac{x}{4} - 1 \\ y &= \frac{x}{4} + 1. \end{aligned}$$

5. Define $g(2)$ in a way that extends the function

$$g(x) = \sin\left(\frac{x^2 - 4x + 4}{x - 2}\right)$$

to be continuous at $x = 2$.

Answer: As given, g is not defined at $x = 2$. However, $h(x) = \sin(x)$ is continuous everywhere and $k(x) = \frac{x^2-4x+4}{x-2}$ is a quotient of polynomials and, hence, continuous everywhere it's defined, meaning everywhere except $x = 2$. Therefore, the limit laws imply that $g(x) = (h \circ k)(x)$ is already continuous everywhere except $x = 2$.

Since g being continuous at $x = 2$ would mean that

$$\lim_{x \to 2} g(x) = g(2),$$

we just need to define $g(2)$ to be $\lim_{x\to 2} g(x)$ in order to get g to be continuous at $x = 2$. Therefore, the problem boils down to computing this limit.

$$\begin{aligned} \lim_{x \to 2} \sin\left(\frac{x^2 - 4x + 4}{x - 2}\right) &= \lim_{x \to 2} \sin\left(\frac{(x-2)^2}{x-2}\right) \\ &= \lim_{x \to 2} \sin(x - 2) \\ &= \sin(2 - 2) \\ &= 0 \end{aligned}$$

where we can pass from the 2nd to the 3rd lines since $\sin(x - 2)$ is continuous everywhere, including at $x = 2$.

Hence, we can extend g to a function which is continuous at $x = 2$ by defining $g(2) = 0$.