Math 2250 Exam #3 Solutions

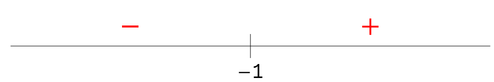
1. Consider the function
$$f(x) = xe^x.$$

   (a) Does $f(x)$ have any asymptotes? If so, what are they? If you claim $f(x)$ has an asymptote, be sure to prove it.

   Answer: $f(x)$ is defined everywhere, so it doesn't have any vertical asymptotes. To check for horizontal asymptotes, we need to evaluate $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. The first is clearly infinity, since both x and e^x go to infinity:
$$\lim_{x\to+\infty} xe^x = +\infty.$$
   On the other hand, in the limit
$$\lim_{x\to-\infty} xe^x$$
   x is going to $-\infty$ and e^x is going to 0, so this is an $\infty \cdot 0$ limit. As usual, we deal with this by dividing by the reciprocal of one of the terms, then using L'Hôpital's Rule:
$$\begin{aligned}\lim_{x\to-\infty} xe^x &= \lim_{x\to-\infty} \frac{x}{e^{-x}}\\ &= \lim_{x\to-\infty} \frac{\frac{d}{dx}(x)}{\frac{d}{dx}(e^{-x})}\\ &= \lim_{x\to-\infty} \frac{1}{-e^{-x}}\\ &= \lim_{x\to-\infty} -e^x\\ &= 0.\end{aligned}$$
   Therefore, we can conclude that the only asymptote for $f(x)$ is a horizontal asymptote at $y = 0$.

   (b) On what interval(s) is $f(x)$ increasing? On what interval(s) is $f(x)$ decreasing? What are the local minima and maxima (if any) of $f(x)$?

   Answer: To answer these questions, we should differentiate $f(x)$:
$$f'(x) = 1 \cdot e^x + xe^x = e^x(1 + x).$$
   Therefore, the only critical point occurs when $f'(x) = 0$, meaning that $x = -1$ (since e^x can never be zero). The sign of $f'(x)$ is



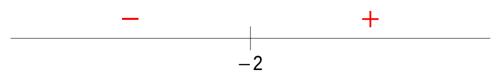
   as we can see by evaluating:
$$\begin{aligned}f'(-2) &= e^{-2}(1 + (-2)) = -\frac{1}{e^2} < 0\\ f'(0) &= e^0(1 + 0) = 1 > 0.\end{aligned}$$
   Therefore, $f(x)$ is decreasing on the interval $(-\infty, -1)$ and increasing on $(-1, +\infty)$. Since $f(x)$ switches from decreasing to increasing at $x = -1$, this must be a local minimum. Since this is the only critical point, there is no local maximum.

(c) On what interval(s) is $f(x)$ concave up? On what interval(s) is $f(x)$ concave down? What are the inflection points (if any) of $f(x)$?

Answer: Now we need to compute the second derivative:

$$f''(x) = e^x \cdot \frac{d}{dx}(1+x) + (1+x) \cdot \frac{d}{dx}(e^x) = e^x + (1+x)e^x = e^x(2+x).$$

Therefore, $f''(x) = 0$ when $x = -2$. The sign of $f''(x)$ is



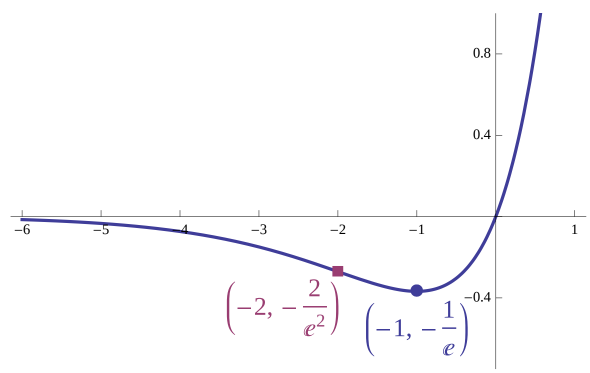
as we can see by evaluating:

$$f''(-3) = e^{-3}(2+(-3)) = -\frac{1}{e^3} < 0$$
$$f''(0) = e^0(2+0) = 2 > 0.$$

Therefore, $f(x)$ is concave down on $(-\infty, -2)$ and concave up on $(-2, +\infty)$. Since the concavity switches there, $x = -2$ is an inflection point.

(d) Use your answers to parts (a)-(c) to sketch the graph of $f(x)$. Make sure your sketch is good enough that concavity is clear, and be sure to label any local or absolute minima/maxima and any inflection points.

Answer: Here's the plot, with the absolute minimum labeled in blue and the inflection point labeled in maroon:



2. Evaluate the following limits:

(a)

$$\lim_{x \to 0} \frac{\tan(x)}{\ln(2x+1)}.$$

Answer: Since the numerator and denominator are both going to zero, this is a perfect opportunity to use L'Hôpital's Rule:

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan(x)}{\ln(2x+1)} &= \lim_{x\to 0} \frac{\frac{d}{dx}(\tan(x))}{\frac{d}{dx}(\ln(2x+1))} \\
&= \lim_{x\to 0} \frac{\sec^2(x)}{\frac{1}{2x+1}\cdot 2} \\
&= \lim_{x\to 0} \frac{\sec^2(x)}{\frac{2}{2x+1}} \\
&= \lim_{x\to 0} \frac{2x+1}{2}\sec^2(x) \\
&= \frac{1}{2}
\end{aligned}$$

since $\sec^2(0) = 1$.

(b)

$$\lim_{x\to 1} x^{1/(x-1)}$$

Answer: We want to use L'Hôpital's Rule again, but now we need to do some manipulation first. As usual when we have a variable base raised to a variable power, we want to take the natural log, evaluate the limit, then exponentiate the answer at the end. To that end, we compute

$$\lim_{x\to 1} \ln\left(x^{1/(x-1)}\right) = \lim_{x\to 1} \frac{1}{x-1}\ln(x) = \lim_{x\to 1} \frac{\ln(x)}{x-1}.$$

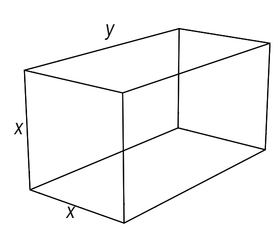
Now we're in a position to use L'Hôpital's Rule since the numerator and denominator are both going to zero.

$$\begin{aligned}
\lim_{x\to 1} \frac{\ln(x)}{x-1} &= \lim_{x\to 1} \frac{\frac{d}{dx}(\ln(x))}{\frac{d}{dx}(x-1)} \\
&= \lim_{x\to 1} \frac{1/x}{1} \\
&= \lim_{x\to 1} \frac{1}{x} \\
&= 1.
\end{aligned}$$

Therefore, we know that the limit we actually want to compute is e raised to this power:

$$\lim_{x\to 1} x^{1/(x-1)} = e^{\lim_{x\to 1}\ln\left(x^{1/(x-1)}\right)} = e^1 = e.$$

3. You have 16 cm of wire with which to build the frame of a box whose base is an $x \times x$ square and whose height is y as shown below. What dimensions maximize the volume of the box having this frame? Be sure to justify your answer.



Answer: Notice that there are 8 segments in the above frame of length x and four of length y, so we know that

$$16 = 8x + 4y,$$

meaning that

$$4y = 16 - 8x$$

or

$$y = 4 - 2x.$$

Now the volume of the box is given by

$$x^2 y = x^2(4 - 2x) = 4x^2 - 2x^3,$$

so our goal is to maximize the function $V(x) = 4x^2 - 2x^3$ subject to the constraint $0 \leq x \leq 2$ (since even if $y = 0$, the fact that we only have 16 cm of wire means that x can't be bigger than 2 cm).

Now we find critical points by differentiating:

$$V'(x) = 8x - 6x^2,$$

so the critical points occur when $V'(x) = 0$, meaning

$$\begin{aligned} 8x - 6x^2 &= 0 \\ x(8 - 6x) &= 0, \end{aligned}$$

so $x = 0$ or $x = \frac{8}{6} = \frac{4}{3}$. Now we just have to evaluate the volume function at the critical points and the endpoints:

$$\begin{aligned} V(0) &= 4(0)^2 - 2(0)^3 = 0 \\ V(4/3) &= 4(4/3)^2 - 2(4/3)^3 = \frac{64}{9} - \frac{128}{27} = \frac{64}{27} \\ V(2) &= 4(2)^2 - 2(2)^3 = 16 - 16 = 0. \end{aligned}$$

Therefore, we get the maximum volume of $\frac{64}{27}\text{cm}^3$ when $x = \frac{4}{3}$, which means that

$$y = 4 - 2 \cdot \frac{4}{3} = 4 - \frac{8}{3} = \frac{4}{3}$$

as well, so the ideal shape of the frame is a cube.

4. In the lab you're growing a colony of *E. coli* which has been genetically modified so that the rate of growth of the colony after t days is proportional to $\sqrt{t}$. In other words, if $P(t)$ is the population of the colony after t days, then

$$P'(t) = k\sqrt{t}$$

for some constant k.

If the initial *E. coli* population consists of 500 bacteria and if you observe 600 bacteria in the colony after 1 day, what is k? Predict the size of the *E. coli* colony after 4 days. (*Hint: you should be able to give this prediction as an explicit integer.*)

Answer: Notice that

$$P(t) = \int P'(t)\, dt = \int k\sqrt{t}\, dt = \int kt^{1/2}\, dt = \frac{2}{3}kt^{3/2} + C$$

for some constant C. Now, we can solve for C by evaluating at $t = 0$:

$$\begin{aligned} P(0) &= 500 \\ \frac{2}{3}k(0)^2 + C &= 500 \\ C &= 500. \end{aligned}$$

Therefore, $P(t) = \frac{2}{3}kt^{3/2} + 500$ is the population of *E. coli* after t days. We can determine the constant k by using the second data point:

$$\begin{aligned} P(1) &= 600 \\ \frac{2}{3}k(1)^2 + 500 &= 600 \\ \frac{2}{3}k + 500 &= 600, \end{aligned}$$

so, after subtracting 500 from both sides, we see that

$$\frac{2}{3}k = 100,$$

meaning that $k = 150$.

Therefore,

$$P(t) = \frac{2}{3} \cdot 150t^{3/2} + 500 = 100t^{3/2} + 500$$

is the population after t days.

In particular, the population after 4 days should be

$$P(4) = 100(4)^{3/2} + 500 = 100(8) + 500 = 800 + 500 = 1300 \text{ bacteria.}$$

5. Find *one* of the local minima of the function

$$g(x) = x^4 - 2x^2 + 2.$$

Make sure you check that your answer really is a local minimum of the function.

Answer: To find a local minimum, the first step is to locate the critical points of the function. Differentiating yields:

$$g'(x) = 4x^3 - 4x = 4x(x^2 - 1).$$

Since this is defined everywhere, the only critical points occur when $g'(x) = 0$, meaning that either $x = 0$ or $x^2 = 1$, so our critical points are

$$x = -1, \qquad x = 0, \qquad x = 1.$$

To test whether a critical point is a local max or a local min, we can use either the first or the second derivative test. In this case, I choose arbitrarily to use the second derivative test. Since

$$f''(x) = 12x^2 - 4,$$

I can see that

$$\begin{aligned} f''(-1) &= 12(-1)^2 - 4 = 12 - 4 = 8 > 0 \\ f''(0) &= 12(0)^2 - 4 = -4 < 0 \\ f''(1) &= 12(1)^2 - 4 = 12 - 4 = 8 > 0, \end{aligned}$$

so the second derivative test tells me that the function has local minima at $x = \pm 1$ and a local maximum at $x = 0$. Therefore, the point

$$(-1, f(-1)) = (-1, 1)$$

is one of the local minima of $g(x)$ (the other is $(1, 1)$).

Not that you had to figure this out, but here's the graph of the function, with the local maximum and local minima labeled: