Math 2250 Exam #3 Practice Problem Solutions

1. Determine the absolute maximum and minimum values of the function

$$f(x) = \frac{x}{1+x^2}.$$

Answer: Notice that f is defined for all x. Also,

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x}{1+x^2} = 0,$$

so f doesn't go off to infinity.

Now, to find the critical points, compute

$$f'(x) = \frac{(1+x^2)(1) - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2},$$

which equals zero precisely when $x^2 = 1$, or $x = \pm 1$. Thus, we just need to evaluate f at the critical points:

$$f(1) = \frac{1}{2}$$
$$f(-1) = -\frac{1}{2}$$

Since f limits to 0 in both directions, we see that these are the absolute maximum and absolute minimum values of the function.

2. Find the inflection points for the function

$$f(x) = 8x + 3 - 2\sin x, \quad 0 < x < 3\pi.$$

Answer: Notice that

$$f'(x) = 8 - 2\cos x$$

and

$$f''(x) = 2\sin x.$$

Now, $\sin x$ changes from positive to negative at $x = \pi$ and from negative to positive at $x = 2\pi$. Since

$$f(\pi) = 8\pi + 3 - 2\sin\pi = 8\pi + 3$$
$$f(2\pi) = 8(2\pi) = 3 - 2\sin 2\pi = 16\pi + 3$$

the inflection points for f between 0 and 3π are

$$(\pi, 8\pi + 3), \quad (2\pi, 16\pi + 3).$$

3. Evaluate the limit

$$\lim_{x\to 0^+} x^2 \csc^2 x.$$

Answer: Re-write the limit as

$$\lim_{x\to 0^+} \frac{x^2}{\sin^2 x}.$$

Since both numerator and denominator go to zero, we can use L'Hôpital's Rule, so this limit equals

$$\lim_{x\to 0^+} \frac{2x}{2\sin x\cos x}.$$

Again, both numerator and denominator go to zero, so apply L'Hôpital's Rule again to get:

$$\lim_{x\to 0^+} \frac{2}{2\cos^2 x - 2\sin^2 x} = \frac{2}{2} = 1.$$

4. Given that

$$f'(t) = 2t - 3\sin t, \quad f(0) = 5,$$

find f.

Answer: We know that

$$f(t) = \int f'(t)dt = \int (2t - 3\sin t)dt = t^2 + 3\cos t + C.$$

Now, since

$$5 = f(0) = 0^2 + 3\cos 0 + C = 3 + C,$$

we see that $C = 2$, so

$$f(t) = t^2 + 3\cos t + 2.$$

5. Find the absolute minimum value of the function

$$f(x) = \frac{e^x}{x}$$

for $x > 0$.

Answer: Notice that

$$\lim_{x\to 0^+} \frac{e^x}{x} = +\infty$$

and

$$\lim_{x\to\infty} \frac{e^x}{x} = \lim_{x\to\infty} \frac{e^x}{1} = \infty$$

by L'Hôpital's Rule. Therefore, we should expect the absolute minimum to occur at a critical point. To find the critical points, take the derivative:

$$f'(x) = \frac{xe^x - e^x}{x^2} = e^x \frac{x-1}{x^2}.$$

This is zero only when $x - 1 = 0$, meaning that f has a single critical point at $x = 1$. Just to double-check that this is indeed the minimum, note that f changes sign from negative to positive at $x = 1$, so, by the first derivative test, f has its minimum at $x = 1$. The minimum value of f is, thus,

$$f(1) = \frac{e^1}{1} = e.$$

6. Evaluate the integral

$$\int \sec 3t \tan 3t dt.$$

Answer: It's easy to check that $\frac{\sec 3t}{3}$ is *an* antiderivative for $\sec 3t \tan 3t$, so

$$\int \sec 3t \tan 3t dt = \frac{\sec 3t}{3} + C.$$

7. Evaluate the limit

$$\lim_{x\to 0} \frac{1 - \cos x}{x^2 + x}.$$

Answer: Notice that both numerator and denominator go to zero as $x \to 0$. Hence, we can apply L'Hopital's Rule:

$$\lim_{x\to 0} \frac{1 - \cos x}{x^2 + x} = \lim_{x\to 0} \frac{\sin x}{2x + 1} = 0,$$

since $\sin(0) = 0$.

8. Find the maximum and minimum values, inflection points and asymptotes of $y = \ln(x^2+1)$ and use this information to sketch the graph.

Answer: Notice that

$$y' = \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1}$$

and, by the Quotient Rule,

$$y'' = \frac{(x^2+1)(2) - 2x(2x)}{(x^2+1)^2} = \frac{2x^2+2-4x^2}{(x^2+1)^2} = \frac{2-2x^2}{(x^2+1)^2}.$$

Now, the critical points occur when $y' = 0$, which is to say when

$$\frac{2x}{x^2+1} = 0.$$

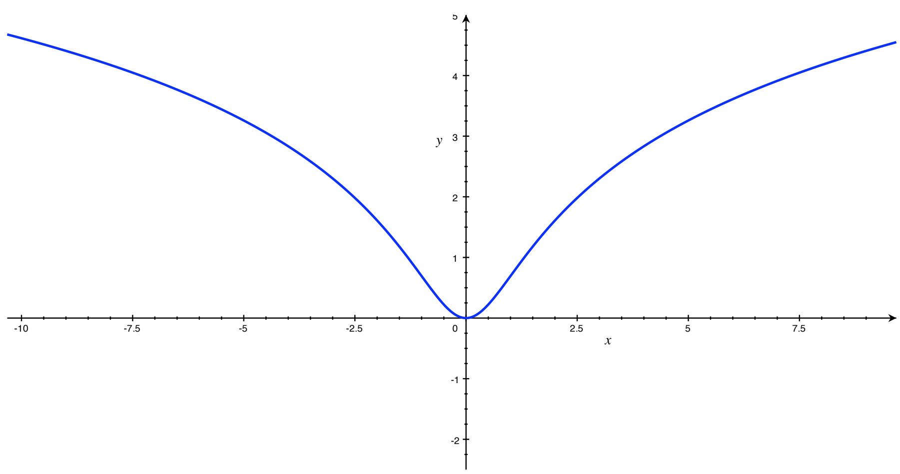
The only happens when $x = 0$, so 0 is the only critical point. Notice that $y''(0) = 2$, which is greater than zero, so the second derivative test implies that 0 is a local minimum.

$y'' = 0$ when $2 - 2x^2 = 0$, meaning when $x = \pm 1$, so there are inflection points at $x = \pm 1$. Finally,

$$\lim_{x \to -\infty} \ln(x^2+1) = \infty = \lim_{x \to +\infty} \ln(x^2+1),$$

so there are no horizontal asymptotes.

Putting all this together, we see that y has a minimum at 0 and is concave up between -1 and 1 and concave down everywhere else and has no asymptotes, meaning that the graph looks something like this:



9. What is the absolute maximum value of $f(x) = x^{1/x}$ for $x > 0$?

Answer: Taking the natural log of both sides,

$$\ln f(x) = \ln(x^{1/x}) = \frac{1}{x} \ln x$$

. Now differentiating, we see that

$$\frac{f'(x)}{f(x)} = \frac{1}{x} \cdot \frac{1}{x} - \frac{1}{x^2} \ln x = \frac{1}{x^2}(1 - \ln x),$$

so

$$f'(x) = f(x)\frac{1}{x^2}(1 - \ln x) = \frac{x^{1/x}}{x^2}(1 - \ln x).$$

Since $x^{1/x}$ is never zero for $x > 0$, $f'(x) = 0$ only when $1 - \ln x = 0$, meaning that $\ln x = 1$. This only happens when $x = e$, so e is the only critical point of f. Notice that $f'(x)$ changes sign from positive to negative at $x = e$, so the first derivative test implies that f has a local maximum at e. However, since this is the only critical point and there are no endpoints, this must, in fact, be the global maximum of f.

Hence, the absolute maximum value of $f(x)$ for $x > 0$ is

$$f(e) = e^{1/e}.$$

10. Suppose the velocity of a particle is given by

$$v(t) = 3\cos t + 4\sin t.$$

If the particle starts (at time 0) at a position 7 units to the right of the origin, what is the position of the particle at time t?

Answer: Let $s(t)$ be the position of the particle at time t. Then we know that $s'(t) = v(t)$ and that $s(0) = 7$. Now,

$$\int v(t)dt = \int (3\cos t + 4\sin t)dt = 3\sin t - 4\cos t + C.$$

Therefore, since $s(t)$ is *an* antiderivative of $v(t) = s'(t)$, we know that

$$s(t) = 3\sin t - 4\cos t + C$$

for some real number C. To solve for C, plug in $t = 0$:

$$7 = s(0) = 3\sin(0) - 4\cos(0) + C = -4 + C,$$

so we see that $C = 11$.

Therefore, the position of the particle is given by

$$s(t) = 3\sin t - 4\cos t + 11.$$

11. Evaluate the limit

$$\lim_{x\to+\infty} x\tan\left(\frac{1}{x}\right).$$

Answer: Notice that, as $x \to +\infty$, $\frac{1}{x}$ goes to zero. Since $\tan(0) = 0$, we see that the above limit takes the form of $\infty \cdot 0$. Therefore, I can convert it into a standard form for applying L'Hôpital's Rule as follows:

$$\lim_{x\to+\infty} x\tan\left(\frac{1}{x}\right) = \lim_{x\to+\infty} \frac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}}.$$

Now, both numerator and denominator go to zero, so L'Hôpital's Rule says that the above limit is equal to

$$\lim_{x\to+\infty} \frac{\sec^2\left(\frac{1}{x}\right)\cdot\frac{-1}{x^2}}{\frac{-1}{x^2}} = \lim_{x\to+\infty} \sec^2\left(\frac{1}{x}\right).$$

In turn, since $\sec\theta = \frac{1}{\cos\theta}$ for any θ, the above limit is equal to

$$\lim_{x\to+\infty} \frac{1}{\cos^2\left(\frac{1}{x}\right)} = \frac{1}{\cos 0} = 1.$$

12. For what value of c does the function $f(x) = x + \frac{c}{x}$ have a local minimum at $x = 3$?

Answer: If f has a local minimum at $x = 3$, then it must be the case that f has a critical point at $x = 3$, meaning that $f'(3) = 0$. Now,

$$f'(x) = 1 - \frac{c}{x^2},$$

so $f'(3) = 0$ implies that

$$1 - \frac{c}{3^2} = 0,$$

or, equivalently,

$$\frac{c}{9} = 1.$$

Hence, f has a critical point at $x = 3$ only if $c = 9$. To double-check that f really has a local minimum here, let $c = 9$ and use the second derivative test. Since

$$f''(x) = \frac{2 \cdot 9}{x^3},$$

we see that $f''(3) = \frac{18}{27} = \frac{2}{3} > 0$, so the second derivative test says that f does indeed have a local minimum at $x = 3$.

13. Draw the graph of the function $g(x) = 4x^3 - x^4$. Label any local maxima or minima, inflection points, and asymptotes, and indicate where the graph is concave up and where it is concave down.

Answer: Notice that

$$\lim_{x \to \pm\infty} \left[4x^3 - x^4\right] = -\infty,$$

so the graph of $g(x)$ has no horizontal asymptotes. Moreover, $g(x)$ is defined for all real numbers, so its graph has no vertical asymptotes.

Now,

$$g'(x) = 12x^2 - 4x^3,$$

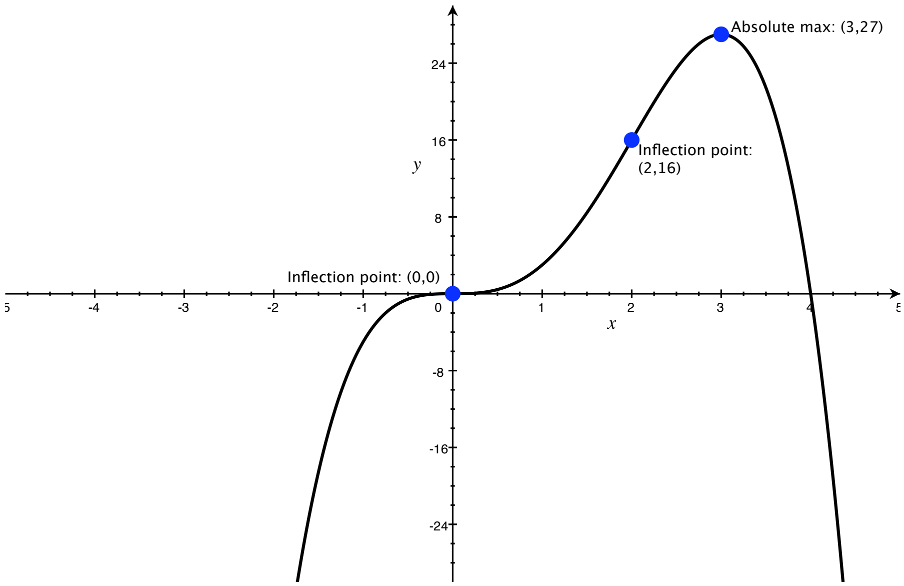
so g has a critical point when

$$0 = 12x^2 - 4x^3 = 4x^2(3 - x).$$

Thus, the critical points of g occur at $x = 0$ and $x = 3$. Take the second derivative:

$$g''(x) = 24x - 12x^2 = 12x(2 - x).$$

Hence, $g''(x) = 0$ when $x = 0$ or $x = 2$. Notice that $g''(x) < 0$ when $x < 0$, that $g''(x) > 0$ when $0 < x < 2$, and $g''(x) < 0$ when $x > 2$. Hence g has inflection points at $x = 0$ and $x = 2$, and the second derivative test tells us that g has a local maximum when $x = 3$ (since $g'(3) = 0$ and $g''(3) < 0$). In fact, this local maximum is an absolute maximum, since $g(x)$ goes to $-\infty$ when $x \to \pm\infty$. Putting this all together yields the following graph



14. Suppose that

$$h'(u) = \frac{u^2+1}{u^2} \quad \text{and that} \quad h(1) = 3.$$

What is $h(2)$?

Answer: Notice that

$$h'(u) = \frac{u^2+1}{u^2} = \frac{u^2}{u^2} + \frac{1}{u^2} = 1 + u^{-2}.$$

Hence,

$$\int h'(u)du = u - u^{-1} + C = u - \frac{1}{u} + C,$$

so

$$h(u) = u - \frac{1}{u} + C$$

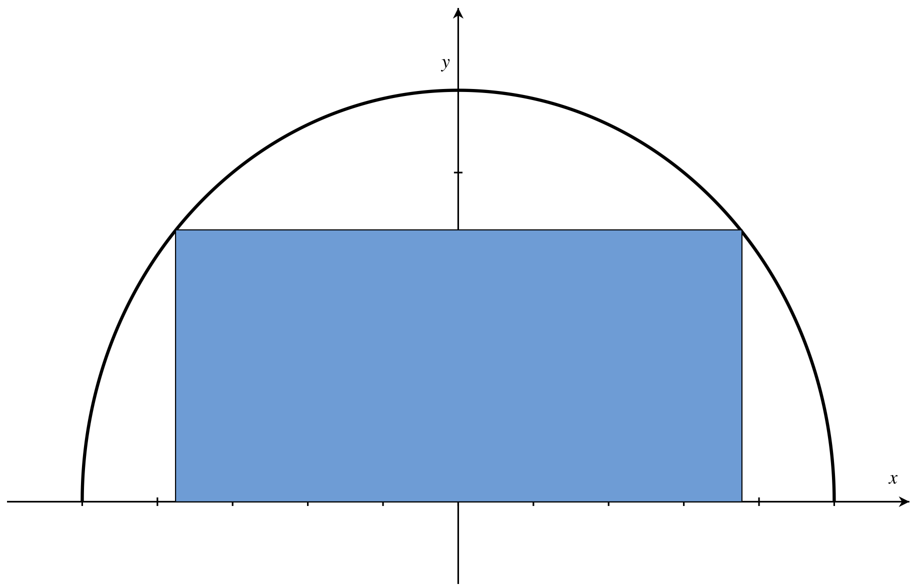
for some real number C. We can solve for C by plugging in $u = 1$:

$$3 = h(1) = 1 - \frac{1}{1} + C = 0 + C = C,$$

so we see that $h(x) = u - \frac{1}{u} + 3$. Therefore,

$$h(2) = 2 - \frac{1}{2} + 3 = \frac{9}{2}.$$

15. A rectangle is bounded by the x-axis and the graph of the function $f(x) = \sqrt{25 - x^2}$ as shown in the figure below. What length and width should the rectangle be so that its area is maximized?



Answer: If the top-right corner of the rectangle is at the point (x, y), then $y = \sqrt{25 - x^2}$ and the four corners of the rectangle will be at the points

$$(-x, 0), (-x, \sqrt{25 - x^2}), (x, \sqrt{25 - x^2}), (x, 0).$$

Hence, the length of the rectangle is $2x$ and the width is $\sqrt{25 - x^2}$. If $A(x)$ is the area of the rectangle, then

$$A(x) = 2x\sqrt{25 - x^2}.$$

Notice that x can be no smaller than 0 and no bigger than 5, so we want to maximize the function $A(x)$ on the interval $[0, 5]$. First, find the critical points of A. The derivative of A is given by

$$A'(x) = 2\sqrt{25 - x^2} + 2x\frac{1}{2}(25 - x^2)^{-1/2}(-2x) = 2\sqrt{25 - x^2} - \frac{2x^2}{\sqrt{25 - x^2}},$$

which, by finding a common denominator, can be simplified to

$$A'(x) = \frac{2(25 - x^2)}{\sqrt{25 - x^2}} - \frac{2x^2}{\sqrt{25 - x^2}} = \frac{50 - 4x^2}{\sqrt{25 - x^2}}.$$

Therefore, $A'(x) = 0$ when

$$0 = 50 - 4x^2$$

or, equivalently, when $x^2 = \frac{50}{4} = \frac{25}{2}$. Therefore, the critical points of A occur when $x = \pm\frac{5}{\sqrt{2}}$. Only the positive one of these is in the interval $[0, 5]$, so we have three points to check: the two endpoints $x = 0$ and $x = 5$ and the critical point $x = \frac{5}{\sqrt{2}}$.

$$A(0) = 2 \cdot 0\sqrt{25 - 0^2} = 0$$
$$A(5/\sqrt{2}) = 2 \cdot \frac{5}{\sqrt{2}}\sqrt{25 - \frac{25}{2}} = 25$$
$$A(5) = 2 \cdot 5\sqrt{25 - 5^2} = 0.$$

Hence, the absolute maximum of the area of the rectangle occurs when $x = \frac{5}{\sqrt{2}}$. This gives a rectangle of length

$$2x = 2 \cdot \frac{5}{\sqrt{2}} = 5\sqrt{2}$$

and width

$$y = \sqrt{25 - x^2} = \sqrt{25 - \frac{25}{2}} = \frac{5}{\sqrt{2}}.$$