

Math 2250 Exam #2 Solutions

1. Find the derivative of each of the following functions.

(a) $f(x) = 2\sqrt{x} \cdot e^{\sqrt{x}}$

Answer: Using the product rule,

$$\begin{aligned}f'(x) &= \frac{d}{dx} (2\sqrt{x}) \cdot e^{\sqrt{x}} + 2\sqrt{x} \cdot \frac{d}{dx} (e^{\sqrt{x}}) \\f'(x) &= 2 \cdot \frac{1}{2\sqrt{x}} \cdot e^{\sqrt{x}} + 2\sqrt{x} \left(e^{\sqrt{x}} \cdot \frac{d}{dx} (\sqrt{x}) \right) \\f'(x) &= \frac{e^{\sqrt{x}}}{\sqrt{x}} + 2\sqrt{x}e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} \\f'(x) &= \frac{e^{\sqrt{x}}}{\sqrt{x}} + e^{\sqrt{x}} \\f'(x) &= e^{\sqrt{x}} \left(\frac{1}{\sqrt{x}} + 1 \right).\end{aligned}$$

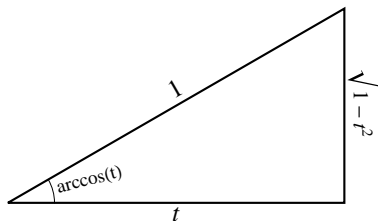
(b) $g(x) = \frac{\ln(x)}{x}$

Answer: Using the quotient rule,

$$\begin{aligned}g'(x) &= \frac{x \cdot \frac{d}{dx}(\ln(x)) - \ln(x) \cdot \frac{d}{dx}(x)}{x^2} \\g'(x) &= \frac{x \cdot \frac{1}{x} - \ln(x) \cdot 1}{x^2} \\g'(x) &= \frac{1 - \ln(x)}{x^2}.\end{aligned}$$

(c) $h(t) = \sin(\arccos(t))$ (simplify your answer as much as possible)

Answer: There are two ways to go here. The first is to recognize that $\sin(\arccos(t)) = \sqrt{1-t^2}$ from the following triangle:



So $h(t) = \sqrt{1-t^2}$ and we can differentiate using the Chain Rule:

$$\begin{aligned}h'(t) &= \frac{d}{dt} (\sqrt{1-t^2}) \\h'(t) &= \frac{1}{2\sqrt{1-t^2}} \cdot \frac{d}{dt} (1-t^2) \\h'(t) &= \frac{1}{2\sqrt{1-t^2}} \cdot (-2t) \\h'(t) &= \frac{-t}{\sqrt{1-t^2}}.\end{aligned}$$

On the other hand, we can apply the Chain Rule directly to the original expression $h(t) = \sin(\arccos(t))$:

$$h'(t) = \cos(\arccos(t)) \cdot \frac{d}{dt}(\arccos(t))$$

$$h'(t) = \cos(\arccos(t)) \cdot \frac{-1}{\sqrt{1-t^2}}$$

$$h'(t) = \frac{-\cos(\arccos(t))}{\sqrt{1-t^2}}$$

$$h'(t) = \frac{-t}{\sqrt{1-t^2}},$$

since $\cos(\arccos(t)) = t$.

As you would hope, we arrive at the same answer either way.

(d) $q(x) = (\ln(x))^x$

Answer: Since we have a variable base raised to a variable power, it's a good idea to use logarithmic differentiation. Taking the natural log of both sides yields

$$\ln(q(x)) = \ln((\ln(x))^x)$$

$$\ln(q(x)) = x \ln(\ln(x)).$$

Now we differentiate both sides:

$$\frac{d}{dx}(\ln(q(x))) = \frac{d}{dx}(x \ln(\ln(x)))$$

$$\frac{1}{q(x)} \cdot q'(x) = \frac{d}{dx}(x) \cdot \ln(\ln(x)) + x \cdot \frac{d}{dx}(\ln(\ln(x)))$$

$$\frac{q'(x)}{q(x)} = 1 \cdot \ln(\ln(x)) + x \cdot \frac{1}{\ln(x)} \cdot \frac{d}{dx}(\ln(x))$$

$$\frac{q'(x)}{q(x)} = \ln(\ln(x)) + x \cdot \frac{1}{\ln(x)} \cdot \frac{1}{x}$$

$$\frac{q'(x)}{q(x)} = \ln(\ln(x)) + \frac{1}{\ln(x)}.$$

Multiplying both sides by $q(x)$ gives that

$$q'(x) = q(x) \left[\ln(\ln(x)) + \frac{1}{\ln(x)} \right].$$

Finally, substituting in $q(x) = (\ln(x))^x$ yields

$$q'(x) = (\ln(x))^x \left[\ln(\ln(x)) + \frac{1}{\ln(x)} \right].$$

- Physicists at CERN discover an amazing particle which seems to violate General Relativity because it can go faster than the speed of light. They determine that when the mysterious particle is produced by a particular hadron collision, its position after t seconds (measured as its distance in meters from the location of the collision) is given by

$$p(t) = \tan(t).$$

What is the acceleration of the particle at time $t = \pi/4$?

Answer: Since the position of the particle is given by $p(t)$, the particle's velocity will be $v(t) = p'(t)$ and its acceleration will be $a(t) = v'(t) = p''(t)$.

Now,

$$p'(t) = \sec^2(t),$$

so

$$p''(t) = \frac{d}{dt}(\sec^2(t)) = 2\sec(t) \cdot \frac{d}{dt}(\sec(t)) = 2\sec(t)(\sec(t)\tan(t)) = 2\sec^2(t)\tan(t).$$

Therefore, the acceleration of the particle after $\pi/4$ seconds is given by

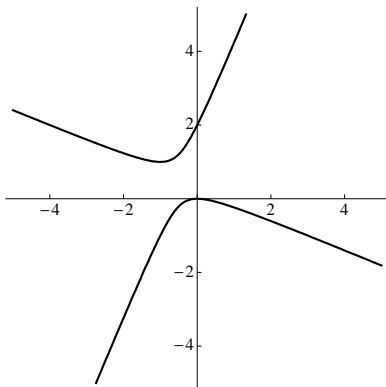
$$p''(\pi/4) = 2\sec^2(\pi/4)\tan(\pi/4) = 2\sqrt{2}^2 \cdot 1 = 4.$$

So the particle is accelerating away from the collision at 4 meters per second per second at this moment.

3. The curve

$$x^2 - y^2 = -2xy - 2y$$

is shown below. At what points does this curve have a horizontal tangent line? (Remember that a point in the plane has two coordinates.)



Answer: The tangent line to a curve is horizontal if and only if the slope of the curve is zero, so our first goal is to compute the slope of the curve. To do so, differentiate implicitly:

$$\begin{aligned} \frac{d}{dx}(x^2 - y^2) &= \frac{d}{dx}(-2xy - 2y) \\ 2x - 2y \frac{dy}{dx} &= \frac{d}{dx}(-2x) \cdot y + (-2x) \cdot \frac{dy}{dx} - 2 \frac{dy}{dx} \\ 2x - 2y \frac{dy}{dx} &= -2y - 2x \frac{dy}{dx} - 2 \frac{dy}{dx}. \end{aligned}$$

Collecting all the $\frac{dy}{dx}$ terms on one side yields

$$2x \frac{dy}{dx} + 2 \frac{dy}{dx} - 2y \frac{dy}{dx} = -2y - 2x,$$

and so, factoring out $2 \frac{dy}{dx}$, we have

$$2 \frac{dy}{dx}(x + 1 - y) = -2y - 2x.$$

We can divide both sides by 2 to get

$$\frac{dy}{dx}(x + 1 - y) = -y - x,$$

so

$$\frac{dy}{dx} = \frac{-y - x}{x + 1 - y}$$

is the slope of the curve at the point (x, y) . Clearly, this is zero if and only if the numerator is zero, so the curve has a horizontal tangent line only if

$$-y - x = 0,$$

or

$$y = -x.$$

So we just need to determine which points on the curve $x^2 - y^2 = -2xy - 2y$ satisfy the equation $y = -x$. To check, just substitute $-x$ for y everywhere it appears in the equation for the curve:

$$\begin{aligned}x^2 - (-x)^2 &= -2x(-x) - 2(-x) \\x^2 - x^2 &= 2x^2 + 2x \\0 &= 2x(x + 1),\end{aligned}$$

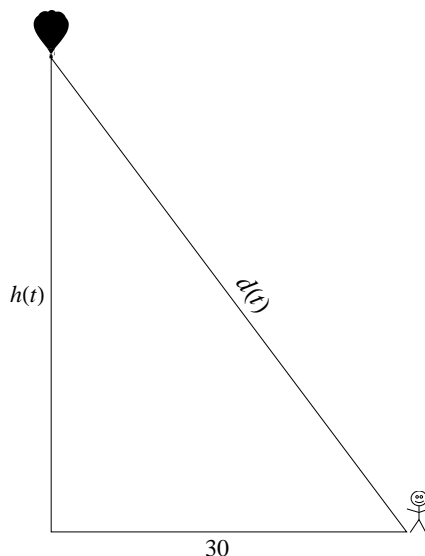
which happens when $x = 0$ or $x = -1$.

These points satisfy the equation $y = -x$. When $x = 0$ we have $y = -0 = 0$, so the first point is $(0, 0)$. When $x = -1$ we have $y = -(-1) = 1$, so the second point is $(-1, 1)$.

Therefore the two points on the curve with horizontal tangents are the origin and the point $(-1, 1)$.

4. A hot-air balloon ascends into the morning air at 5 meters per second. You're standing 30 meters (on flat ground) from the spot where the balloon was released. How fast is the distance between you and the balloon changing when the balloon is 40 meters above the ground?

Answer: First, let's draw a picture of the situation:



Here I'm letting $h(t)$ be the height of the balloon above the ground, and I'm letting $d(t)$ be the distance from you to the balloon. Then by the Pythagorean Theorem we know that

$$d(t)^2 = 30^2 + h(t)^2.$$

Also, we know that $h'(t) = 5$ and that, at the specific time t_0 that we're interested in, $h(t_0) = 40$.

Now, we're trying to figure out $d'(t_0)$. Notice that, if we differentiate the equation $d(t)^2 = 30^2 + h(t)^2$ with respect to t , then we have

$$2d(t)d'(t) = 2h(t)h'(t).$$

Therefore,

$$d'(t) = \frac{2h(t)h'(t)}{2d(t)} = \frac{h(t)h'(t)}{d(t)}$$

for any time t . In particular,

$$d'(t_0) = \frac{h(t_0)h'(t_0)}{d(t_0)}.$$

Since $h(t_0) = 40$, Pythagorean Theorem tells us that

$$d(t_0) = \sqrt{30^2 + h(t_0)^2} = \sqrt{30^2 + 40^2} = \sqrt{900 + 1600} = \sqrt{2500} = 50.$$

Therefore, we can just plug $h(t_0) = 40$, $h'(t_0) = 5$, and $d(t_0) = 50$ into the expression $d'(t_0) = \frac{h(t_0)h'(t_0)}{d(t_0)}$ to see that

$$d'(t_0) = \frac{40 \cdot 5}{50} = 4.$$

So the distance from you to the ballon is increasing at a rate of 4 meters per second at this moment.

5. (5 points) You are taking a calculus exam and your mean-spirited calculus professor doesn't allow you to use calculators on exams. To solve one of the problems you need to compute $\sqrt{4.04}$, but of course this is tricky with no calculator. Use your knowledge of calculus to approximate $\sqrt{4.04}$ and save the day.

Answer: The solution, of course, is to use a linearization to approximate $\sqrt{4.04}$. Let $f(x) = \sqrt{x}$. Then $\sqrt{4.04} = f(4.04)$. Since 4.04 is very close to 4 and it's easy to take the square root of 4, we will approximate $\sqrt{4.04}$ by evaluating the linearization of $f(x)$ at $x = 4$. By definition, this linearization is

$$L(x) = f(4) + f'(4)(x - 4).$$

Now, $f(4) = \sqrt{4} = 2$. Also, since $f'(x) = \frac{1}{2\sqrt{x}}$, we see that $f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$. Therefore, the linearization of $f(x)$ at $x = 4$ is

$$L(x) = 2 + \frac{1}{4}(x - 4).$$

Evaluating at 4.04 yields

$$L(4.04) = 2 + \frac{1}{4}(4.04 - 4) = 2 + \frac{1}{4}(0.04) = 2 + 0.01 = 2.01$$

and so $\sqrt{4.04} \approx 2.01$.

(Mathematica tells me that $\sqrt{4.04} = 2.009975\dots$, so the above approximation is pretty good.)