Math 2250 Exam #2 Practice Problem Solutions

1. Let $u(x) = \sqrt{f(x)}$ and suppose $f(3) = 1$, $f'(3) = 8$, and $f''(3) = -2$. What is the value of $u''(3)$?

   Answer: Using the Chain Rule,
$$u'(x) = \frac{1}{2}(f(x))^{-1/2} f'(x).$$
   Hence, using the Chain Rule and the Product Rule,
$$u''(x) = \frac{1}{2}\left(-\frac{1}{2}(f(x))^{-3/2}f'(x)f'(x) + (f(x))^{-1/2}f''(x)\right) = -\frac{(f'(x))^2}{4(f(x))^{3/2}} + \frac{f''(x)}{2\sqrt{f(x)}}.$$
   Therefore,
$$u''(3) = -\frac{(f'(3))^2}{4(f(3))^{3/2}} + \frac{f''(3)}{\sqrt{f(3)}} = -\frac{8^2}{4(1)^{3/2}} + \frac{-2}{2\sqrt{1}} = -17.$$

2. Let $f(x) = \frac{1}{2}\sin(x^2)\cos(x^2)$. What is $f'\left(\sqrt{\frac{5\pi}{6}}\right)$?

   Answer: Using the Product and Chain Rules,
$$f'(x) = \frac{1}{2}\left(\cos(x^2)\cdot 2x\cdot\cos(x^2) + \sin(x^2)\cdot 2x\cdot(-\sin(x^2))\right) = x\cos^2(x^2) - x\sin^2(x^2) = x\left(\cos^2(x^2) - \sin^2(x^2)\right).$$
   Therefore,
$$\begin{aligned} f'\left(\sqrt{\frac{5\pi}{6}}\right) &= \sqrt{\frac{5\pi}{6}}\left(\cos^2\left(\frac{5\pi}{6}\right) - \sin^2\left(\frac{5\pi}{6}\right)\right) \\ &= \sqrt{\frac{5\pi}{6}}\left(\left(-\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right) \\ &= \sqrt{\frac{5\pi}{6}}\left(\frac{3}{4} - \frac{1}{4}\right) \\ &= \frac{1}{2}\sqrt{\frac{5\pi}{6}}. \end{aligned}$$

3. What is the tangent line to $x^3 + y^3 = 6xy$ at $(3, 3)$?

   Answer: Differentiating both sides of the equation with respect to x yields
$$3x^2 + 3y^2\frac{dy}{dx} = 6\left(y + x\frac{dy}{dx}\right).$$
   Re-arranging gives that
$$\left(3y^2 - 6x\right)\frac{dy}{dx} = 6y - 3x^2,$$
   so
$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}.$$
   Plugging in $(x, y) = (3, 3)$, we see that the slope of the tangent line at this point is
$$\frac{6(3) - 3(3)^2}{3(3)^2 - 6(3)} = \frac{18 - 27}{27 - 18} = \frac{-9}{9} = -1.$$
   Therefore, using the point-slope formula, the equation of the tangent line is
$$y - 3 = -1(x - 3),$$
   or
$$y = -x + 6.$$

4. Suppose $y = x^{\arcsin(x)}$. What is $\frac{dy}{dx}$?

   Answer: We will find the derivative using logarithmic differentiation. Thus, we first take the natural log of both sides:

$$\ln y = \ln(x^{\arcsin(x)}) = \arcsin(x)\ln(x).$$

   Now, differentiating both sides with respect to x, we see that

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \cdot \ln(x) + \arcsin(x)\frac{1}{x}$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{\ln(x)}{\sqrt{1-x^2}} + \frac{\arcsin(x)}{x}$$

   Multiplying both sides by y gives us that

$$\frac{dy}{dx} = y\left(\frac{\ln(x)}{\sqrt{1-x^2}} + \frac{\arcsin(x)}{x}\right)$$

   so, plugging in $y = x^{\arcsin(x)}$ we see that

$$\frac{dy}{dx} = x^{\arcsin(x)}\left(\frac{\ln(x)}{\sqrt{1-x^2}} + \frac{\arcsin(x)}{x}\right).$$

5. Use a linearization of an appropriate function to estimate $\ln(0.9)$.

   Answer: We will approximate $\ln(0.9)$ using the linearization of $f(x) = \ln(x)$ at 1. This linearization is given by

$$L(x) = f(1) + f'(1)(x-1).$$

   Since $f'(x) = \frac{1}{x}$, we have that

$$f(1) = \ln(1) = 0$$
$$f'(1) = \frac{1}{1} = 1.$$

   Plugging these values into the expression for $L(x)$ yields

$$L(x) = 0 + (x-1) = x - 1.$$

   Therefore,

$$\ln(0.9) \approx L(0.9) = 0.9 - 1 = -0.1.$$

6. The equation

$$(4-x)y^2 = x^3,$$

   determines a curve called a *cissoid*, pictured below. What is the equation of the tangent line to the cissoid at the point $(2,2)$?

   Answer: Differentiating the above equation on both sides yields

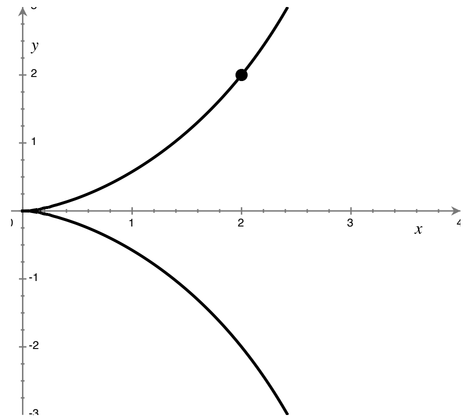
$$-1 \cdot y^2 + (4-x) \cdot 2yy' = 3x^2$$

   or, equivalently,

$$(4-x) \cdot 2yy' = 3x^2 + y^2.$$

   Solving for y' yields

$$y' = \frac{3x^2 + y^2}{2y(4-x)}.$$



Therefore, at the point $(2, 2)$, the slope of the tangent line is given by plugging in $(x, y) = (2, 2)$ in the above equation:

$$\text{Slope} = \frac{3(2)^2 + 2^2}{2(2)(2)} = \frac{16}{8} = 2.$$

Therefore, the slope of the tangent line is 2, and so, by the point-slope formula, the tangent line is given by

$$y - 2 = 2(x - 2)$$

or, equivalently,

$$y = 2x - 2.$$

7. Consider the function

$$f(x) = \sqrt[5]{\sin x}.$$

At which values of x does the graph of f have a vertical tangent line?

Answer: A line is vertical when its slope is infinite (either $+\infty$ or $-\infty$). Since the slope of the tangent line to the graph is given by the derivative, the tangent line will be vertical when the derivative approaches $\pm\infty$. Now, we compute the derivative using the Chain Rule:

$$f'(x) = \frac{1}{5}(\sin x)^{-4/5} \cdot \cos x = \frac{\cos x}{5(\sin x)^{4/5}}.$$

Clearly, this approaches $\pm\infty$ when $\sin x = 0$, so the graph of the function will have a vertical tangent line whenever $\sin x = 0$, which happens when

$$x = n\pi \text{ for any integer } n.$$

8. Estimate $\tan(0.05)$ using an appropriate linearization.

Answer: Since 0.05 is close to zero and we know that

$$\tan(0) = \frac{\sin(0)}{\cos(0)} = \frac{0}{1} = 0,$$

it makes sense to approximate $\tan(0.05)$ by plugging in 0.05 to the linearization of $f(x) = \tan(x)$ at zero. By definition, this linearization is

$$L(x) = f(0) + f'(0)(x - 0).$$

As already indicated, $f(0) = 0$. Now, $f'(x) = \sec^2 x = \frac{1}{\cos x}$, so

$$f'(0) = \frac{1}{(\cos 0)^2} = \frac{1}{1^1} = 1.$$

Therefore, the linearization is

$$L(x) = 0 + 1(x - 0),$$

or $L(x) = x$. Therefore,

$$\tan(0.05) \approx L(0.05) = 0.05.$$

9. A spherical balloon is inflated by an electric pump. To prevent strain on the material, you want to inflate the balloon in such a way that the surface area is increasing at a constant rate of 20 square feet per minute. At what rate (in cubic feet per minute) should air be pumped into the balloon when the radius of the balloon is 2 feet?

Answer: We know that

$$V(t) = \frac{4}{3}\pi\,(r(t))^3$$

and

$$A(t) = 4\pi\,(r(t))^2\,.$$

The statement of the problem tells us that $A'(t) = 20$ for all t, and that, at the time t_0 we're interested in, $r(t_0) = 2$.

We're asked to determine the rate of change of the volume of the balloon at time t_0, which is just $V'(t_0)$. Notice that

$$V'(t) = \frac{4}{3}\pi \cdot 3(r(t))^2 r'(t) = 4\pi(r(t))^2 r'(t).$$

Therefore,

$$V'(t_0) = 4\pi(r(t_0))^2 r'(t_0).$$

Substituting in $r(t_0) = 2$ gives that

$$V'(t_0) = 4\pi(2)^2 r'(t_0) = 16\pi r'(t_0),$$

so the problem boils down to determining $r'(t_0)$.

Since we know $A'(t_0)$ and since $r'(t_0)$ will appear in the expression for $A'(t_0)$, to do so we need to differentiate $A(t)$:

$$A'(t) = 4\pi \cdot 2r(t)r'(t) = 8\pi r(t)r'(t).$$

Therefore, at time t_0,

$$20 = A'(t_0) = 8\pi r(t_0)r'(t_0) = 8\pi \cdot 2r'(t_0) = 16\pi r'(t_0),$$

so we can solve for $r'(t_0)$:

$$r'(t_0) = \frac{20}{16\pi} = \frac{5}{4\pi}.$$

Therefore, using the expression for $V'(t_0)$ given above, we have that

$$V'(t_0) = 16\pi r'(t_0) = 16\pi\frac{5}{4\pi} = 20,$$

so we should be pumping air in at 20 cubic feet per minute when the radius of the balloon is 2 feet.

10. An asteroid hits the Atlantic Ocean and creates an expanding circular wave. If the area enclosed by this wave increases at the rate of 200 km^2/min, how fast is the *diameter* of the wave expanding when its *radius* is 20 km?

Answer: We know that the area enclosed by the wave is given by $A(t) = \pi r(t)^2$ and that the diameter is $2r(t)$. From this latter, the rate of change of the diameter (which is what we're trying to determine) is $2r'(t)$.

Differentiating the expression for A,

$$A'(t) = 2\pi r(t) r'(t).$$

Therefore,

$$r'(t) = \frac{A'(t)}{2\pi r(t)}.$$

At the time t_0 that we're interested in, $A'(t_0) = 200$ and $r(t_0) = 20$. Plugging these values into the above equation, we see that

$$r'(t_0) = \frac{200}{2\pi(20)} = \frac{200}{40\pi} = \frac{5}{\pi}.$$

Therefore, at this instant the diameter is increasing at a rate of

$$2r'(t_0) = 2\frac{5}{\pi} = \frac{10}{\pi} \text{ km/min}.$$

11. Find the equation of the tangent line to the graph of $2e^{xy} = x + y$ at the point $(0, 2)$.

Answer: We use implicit differentiation to take the derivative of both sides with respect to x:

$$\begin{aligned} \frac{d}{dx}\left(2e^{xy}\right) &= \frac{d}{dx}(x+y) \\ 2e^{xy}\frac{d}{dx}(xy) &= 1 + \frac{dy}{dx} \\ 2e^{xy}\left(x\frac{dy}{dx} + y\right) &= 1 + \frac{dy}{dx}, \end{aligned}$$

where I used the Chain Rule to go from the first line to the second on the left hand side.

Therefore, collecting terms, we have

$$\left(2xe^{xy} - 1\right)\frac{dy}{dx} = 1 - 2ye^{xy}$$

or, equivalently,

$$\frac{dy}{dx} = \frac{1 - 2ye^{xy}}{2xe^{xy} - 1}.$$

Therefore, the slope of the tangent line at the point $(0, 2)$ is

$$\frac{1 - 2(2)e^{0\cdot 2}}{2(0)e^{0\cdot 2} - 1} = \frac{1-4}{-1} = 3.$$

Hence, using the point-slope formula, the equation of the tangent line is

$$y - 2 = 3(x - 0),$$

or, equivalently,

$$y = 3x + 2$$

12. At what x values does the graph of the function $f(x) = \tan(1 - x^2)$ have a horizontal tangent line?

    Answer: The graph will have a horizontal tangent at all values of x for which $f'(x) = 0$. Now,

$$f'(x) = \sec^2(1 - x^2) \cdot (-2x) = -2x \sec^2(1 - x^2).$$

Since

$$\sec^2 \theta = \frac{1}{\cos^2 \theta}$$

is never zero for any value of θ, we see that $f'(x) = 0$ only if $x = 0$, so the graph of $f(x)$ has a horizontal tangent line only at $x = 0$.

13. Let $y = x^{\ln(x)}$. What is $\frac{dy}{dx}$?

    Answer: First, take the natural log of both sides:

$$\ln(y) = \ln\left(x^{\ln(x)}\right) = \ln(x)\ln(x).$$

Now, differentiating both sides with respect to x, we see that

$$\frac{1}{y}\frac{dy}{dx} = \ln(x)\frac{1}{x} + \frac{1}{x}\ln(x) = 2\frac{\ln(x)}{x}.$$

Therefore,

$$\frac{dy}{dx} = 2y\frac{\ln(x)}{x}.$$

Plugging in $y = x^{\ln(x)}$, we see that

$$\frac{dy}{dx} = 2x^{\ln(x)}\frac{\ln(x)}{x}.$$