Math 2250 Exam #1 Solutions

1. Evaluate the following limits or explain why they don't exist.

(a)

$$\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h}.$$

Answer: Notice that both the numerator and the denominator are going to zero, so we need to think a little bit. In this case, the most productive thing to do is to rationalize the numerator:

$$\begin{aligned}\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h} &= \lim_{h\to 0} \frac{\sqrt{1+h}-1}{h} \cdot \frac{\sqrt{1+h}+1}{\sqrt{1+h}+1} \\ &= \lim_{h\to 0} \frac{(1+h)-1}{h\left(\sqrt{1+h}+1\right)} \\ &= \lim_{h\to 0} \frac{h}{h\left(\sqrt{1+h}+1\right)} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{1+h}+1} \\ &= \frac{1}{2}.\end{aligned}$$

(b)

$$\lim_{t\to 1} |\sin(t-1)|.$$

Answer: Notice, first of all, that the function $\sin(t-1)$ is a continuous function, so we can evaluate $\lim_{t\to 1}\sin(t-1)$ just by plugging in $t = 1$. Taking absolute values doesn't affect continuity, so in fact

$$\lim_{t\to 1} |\sin(t-1)| = |\sin(1-1)| = |\sin(0)| = 0.$$

(c)

$$\lim_{x\to 2} \sqrt{x^2-4}.$$

Answer: So long as $x \geq 2$, then $x^2 \geq 4$ and $\sqrt{x^2-4}$ makes sense. Indeed,

$$\lim_{x\to 2^+} \sqrt{x^2-4} = 0.$$

However, when $|x| < 2$, we have that $x^2 - 4 < 0$, so $\sqrt{x^2-4}$ does not make sense. In particular, this means that

$$\lim_{x\to 2^-} \sqrt{x^2-4} \text{ does not exist.}$$

Therefore, we can only conclude that $\lim_{x\to 2}\sqrt{x^2-4}$ doesn't exist, either.

(d)

$$\lim_{x\to 1^-} \frac{1}{(x-1)^3}$$

Answer: Notice that for x close to 1 the denominator gets very close to 0. Since the numerator is always 1, we can see that the limit will be $\pm\infty$. To determine which, let's figure out the sign of the denominator.

When $x < 1$, we know that $x - 1 < 0$, which means that $(x-1)^3 < 0$. Therefore, the denominator is always negative. Since the numerator is positive, this means that the limit must be $-\infty$:

$$\lim_{x\to 1^-} \frac{1}{(x-1)^3} = -\infty.$$

2. Let

$$h(x) = \frac{x^2 + x - 6}{3x^2 - 6x}.$$

What are the vertical and horizontal asymptotes of the curve $y = h(x)$?

Answer: To find the horizontal asymptotes, we need to evaluate

$$\lim_{x\to+\infty} h(x) \quad \text{and} \quad \lim_{x\to-\infty} h(x).$$

For the first, notice that the highest power of x that we see is x^2, so we can divide numerator and denominator by x^2:

$$\begin{aligned}
\lim_{x\to+\infty} \frac{x^2 + x - 6}{3x^2 - 6x} &= \lim_{x\to+\infty} \frac{x^2 + x - 6}{3x^2 - 6x} \cdot \frac{1/x^2}{1/x^2} \\
&= \lim_{x\to+\infty} \frac{\frac{1}{x^2}\left(x^2 + x - 6\right)}{\frac{1}{x^2}\left(3x^2 - 6x\right)} \\
&= \lim_{x\to+\infty} \frac{1 + \frac{1}{x} - \frac{6}{x^2}}{3 - \frac{6}{x}} \\
&= \frac{1}{3}.
\end{aligned}$$

Therefore, $h(x)$ has a horizontal asymptote at $y = \frac{1}{3}$.

Evaluating the limit as $x \to -\infty$ is the same calculation:

$$\begin{aligned}
\lim_{x\to-\infty} \frac{x^2 + x - 6}{3x^2 - 6x} &= \lim_{x\to-\infty} \frac{x^2 + x - 6}{3x^2 - 6x} \cdot \frac{1/x^2}{1/x^2} \\
&= \lim_{x\to-\infty} \frac{\frac{1}{x^2}\left(x^2 + x - 6\right)}{\frac{1}{x^2}\left(3x^2 - 6x\right)} \\
&= \lim_{x\to-\infty} \frac{1 + \frac{1}{x} - \frac{6}{x^2}}{3 - \frac{6}{x}} \\
&= \frac{1}{3}.
\end{aligned}$$

So $h(x)$ only has the one horizontal asymptote $y = \frac{1}{3}$.

To find the vertical asymptotes, we first find the values of x which are not in the domain of h. Clearly, the only places where h is undefined are those values of x for which the denominator is equal to zero:

$$0 = 3x^2 - 6x = 3x(x - 2).$$

So the possible vertical asymptotes are $x = 0$ and $x = 2$. Now, we need to check that the function really shoots off to infinity at these places.

First of all,

$$\lim_{x\to0^+} \frac{x^2 + x - 6}{3x^2 - 6x} = +\infty$$

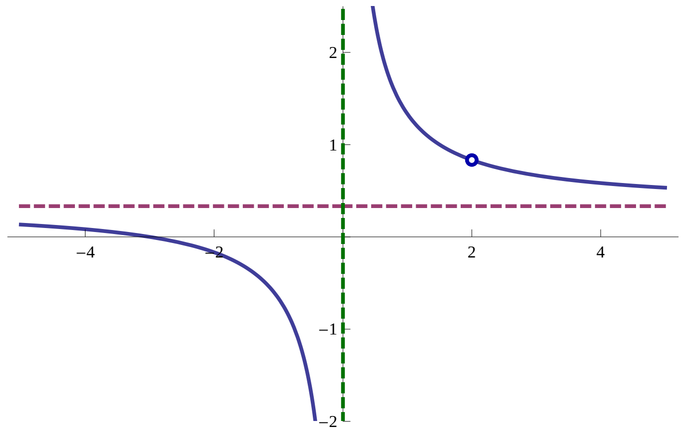
since the numerator is going to -6 while the denominator is negative and going to zero. So $h(x)$ has a vertical asymptote at $x = 0$. (We could also have seen this by evaluating $\lim_{x\to0^-} h(x) = -\infty$).

On the other hand,

$$\lim_{x\to2} \frac{x^2 + x - 6}{3x^2 - 6x} = \lim_{x\to2} \frac{(x+3)(x-2)}{3x(x-2)} = \lim_{x\to2} \frac{x+3}{3x} = \frac{5}{6},$$

so $h(x)$ does *not* have a vertical asymptote at $x = 2$; it just has a removable discontinuity at $x = 2$.

Therefore we conclude that the function $h(x)$ has a single horizontal asymptote at $y = \frac{1}{3}$ and a single vertical asymptote at $x = 0$. Indeed, here's the graph of $h(x)$ with the two asymptotes shown:



3. The position of a ball dropped from a height 50 m above the ground at a time t seconds after it is released is given (approximately) by the function

$$p(t) = -5t^2 + 50.$$

(a) What is the average velocity of the ball (i.e., the average rate of change of the position of the ball) between 1 and 3 seconds after it is released?

Answer: As always, the average rate of change is given by

$$\frac{\text{change of output}}{\text{change of input}} = \frac{p(3) - p(1)}{3 - 1} = \frac{(-5(3)^2 + 50) - (-5(1)^2 + 50)}{2} = \frac{-45 + 5}{2} = \frac{-40}{2} = -20.$$

So the average velocity between $t = 1$ and $t = 3$ is -20m/s.

(b) What is the average velocity of the ball between time 1 and time T? (*Hint:* When $T = 3$ your answer to this question should agree with your answer to part (3a).)

Answer: Again, the average rate of change is given by

$$\frac{\text{change of output}}{\text{change of input}} = \frac{p(T) - p(1)}{T - 1} = \frac{(-5T^2 + 50) - (-5(1)^2 + 50)}{T - 1} = \frac{-5T^2 + 5}{T - 1}.$$

We can simplify further by noting that the numerator factors as $-5(T^2 - 1) = -5(T + 1)(T - 1)$:

$$\frac{-5T^2 + 5}{T - 1} = \frac{-5(T + 1)(T - 1)}{T - 1} = -5(T + 1)$$

so long as $T \neq 1$.

(c) What is the instantaneous velocity of the ball (i.e., the instantaneous rate of change of the position of the ball) exactly 1 second after it is released? (*Hint*: If you haven't already, it may be helpful to simplify your answer to (3b) as much as possible.)

Answer: The instantaneous rate of change at $t = 1$ is just the limit of the average rate of change between 1 and T as we let $T \to 1$, so the instantaneous rate of change is

$$\lim_{T \to 1} \frac{p(T) - p(1)}{T - 1} = \lim_{T \to 1} -5(T + 1) = -10,$$

using the result from part (3b).

Therefore, the velocity of the ball 1 second after it is released is -10m/s.

4. (a) Is the function $p(x) = \lfloor x \rfloor^2$ continuous at $x = 0$? Explain why or why not. (Remember that $\lfloor x \rfloor$, called the *floor* of x, is the biggest integer $\leq x$. For example, $\lfloor 2 \rfloor = 2$, $\lfloor 5.9 \rfloor = 5$, $\lfloor -3.14 \rfloor = -4$, etc.)

Answer: Remember that $p(x)$ is continuous at $x = 0$ if and only if

$$\lim_{x \to 0} p(x) = p(0).$$

Now certainly $p(0) = \lfloor 0 \rfloor^2 = 0^2 = 0$, so the question is whether or not

$$\lim_{x \to 0} p(x) = 0.$$

For $0 < x < 1$, we know that $p(x) = \lfloor x \rfloor^2 = 0^2 = 0$, so it must be the case that

$$\lim_{x \to 0^+} p(x) = 0.$$

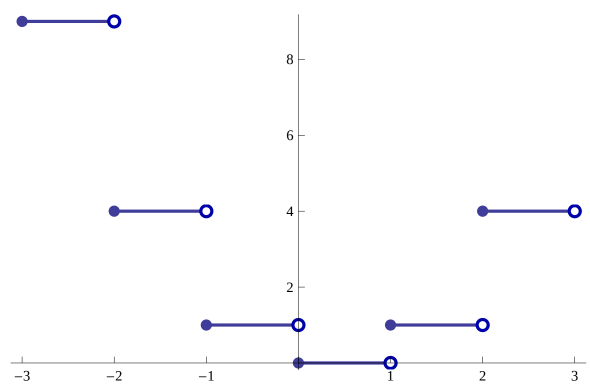
On the other hand, for $-1 < x < 0$, we have that $p(x) = \lfloor x \rfloor^2 = (-1)^2 = 1$, so

$$\lim_{x \to 0^-} p(x) = 1.$$

Therefore,

$$\lim_{x \to 0^+} p(x) \neq \lim_{x \to 0^-} p(x),$$

so in fact $\lim_{x \to 0} p(x)$ does not exist. In particular, this means that $p(x)$ *cannot* be continuous at $x = 0$. Here's the graph of $p(x)$, which is clearly not continuous at $x = 0$:



(b) Is the function $q(x) = \lfloor x^2 \rfloor$ continuous at $x = 0$? Explain why or why not.

Answer: As above, the question is whether or not

$$\lim_{x \to 0} q(x) = q(0).$$

Again, the right hand side is easy to evaluate: $q(0) = \lfloor 0^2 \rfloor = \lfloor 0 \rfloor = 0$.

As for the limit, we again deal with the one-sided limits separately. For $0 < x < 1$, we know that $0 < x^2 < 1$, so $q(x) = \lfloor x^2 \rfloor = 0$, and hence

$$\lim_{x \to 0^+} q(x) = 0.$$

On the other hand, for $-1 < x < 0$, we have $0 < x^2 < 1$, so $q(x) = \lfloor x^2 \rfloor = 0$ and

$$\lim_{x \to 0^-} q(x) = 0.$$

This time the two one-sided limits are both zero, which means that

$$\lim_{x \to 0} q(x) = 0$$

as well. Since $q(0) = 0$, this implies that $q(x)$ *is* continuous at $x = 0$. Here's the graph of $q(x)$: