Math 2250 Exam #1 Practice Problem Solutions

1. Find the vertical asymptotes (if any) of the functions

$$g(x) = 1 + \frac{2}{x}, \qquad h(x) = \frac{4x}{4 - x^2}.$$

Answer: The only number not in the domain of g is $x = 0$, so the only place where g could possibly have a vertical asymptote is at $x = 0$. Since g will have a vertical asymptote at $x = 0$ if either $\lim_{x\to 0^+} g(x)$ or $\lim_{x\to 0^-} g(x)$ is infinite. But clearly

$$\lim_{x\to 0^+}\left(1 + \frac{2}{x}\right) = +\infty,$$

so we know that g does indeed have a vertical asymptote at $x = 0$.

The only numbers not in the domain of h are those x so that

$$0 = 4 - x^2 = (2 - x)(2 + x),$$

so the only numbers not in the domain of h are $x = \pm 2$. Since

$$\lim_{x\to -2^+} h(x) = \lim_{x\to -2^+} \frac{4x}{4 - x^2} = +\infty$$

and

$$\lim_{x\to 2^+} h(x) = \lim_{x\to 2^+} \frac{4x}{4 - x^2} = -\infty,$$

the function h has vertical asymptotes when at both $x = -2$ and $x = 2$.

2. Evaluate

$$\text{(a)} \lim_{x\to 2} \frac{x^2 - 4}{x^2 - 5x + 6} \qquad \text{(b)} \lim_{x\to -2} \frac{|x+2|}{x+2} \qquad \text{(c)} \lim_{x\to\infty} \frac{4x^3 + 2x - 4}{4x^2 - 5x + 6x^3}$$

(a) We can factor the numerator as

$$x^2 - 4 = (x+2)(x-2)$$

and the denominator as

$$x^2 - 5x + 6 = (x-2)(x-3).$$

Therefore,

$$\lim_{x\to 2} \frac{x^2 - 4}{x^2 - 5x + 6} = \lim_{x\to 2} \frac{(x+2)(x-2)}{(x-2)(x-3)} = \lim_{x\to 2} \frac{x+2}{x-3} = \frac{4}{-1} = -4.$$

(b) When $x < -2$, the quantity $x + 2$ is negative, so

$$|x+2| = -(x+2).$$

Hence,

$$\lim_{x\to -2^-} \frac{|x+2|}{x+2} = \lim_{x\to -2^-} \frac{-(x+2)}{x+2} = -1.$$

On the other hand, when $x > -2$, the quantity $x + 2$ is positive, so

$$|x+2| = x + 2.$$

Therefore,

$$\lim_{x\to -2^+} \frac{|x+2|}{x+2} = \lim_{x\to -2^+} \frac{x+2}{x+2} = 1.$$

Since the limits from the left and right don't agree,

$$\lim_{x \to -2} \frac{|x+2|}{x+2}$$

does not exist.

(c) Dividing numerator and denominator by x^3, we get that

$$\begin{aligned}
\lim_{x\to\infty} \frac{4x^3+2x-4}{4x^2-5x+6x^3} &= \lim_{x\to\infty} \frac{\frac{1}{x^3}\left(4x^3+2x-4\right)}{\frac{1}{x^3}\left(4x^2-5x+6x^3\right)} \\
&= \lim_{x\to\infty} \frac{4+\frac{2}{x^2}-\frac{4}{x^3}}{\frac{4}{x}-\frac{5}{x^2}+6} \\
&= \frac{4}{6} \\
&= \frac{2}{3}.
\end{aligned}$$

3. Evaluate

$$\lim_{x\to 6} \frac{x^2-36}{3x^2-16x-12}$$

Answer: The numerator factors as

$$x^2-36 = (x+6)(x-6),$$

while the denominator factors as

$$3x^2-16x-12 = (3x+2)(x-6).$$

Therefore,

$$\lim_{x\to 6} \frac{x^2-36}{3x^2-16x-12} = \lim_{x\to 6} \frac{(x+6)(x-6)}{3x+2)(x-6)} = \lim_{x\to 6} \frac{x+6}{3x+2} = \frac{12}{20} = \frac{3}{5}$$

4. Evaluate

$$\lim_{x\to\infty} \frac{\sqrt[3]{x^2-3x+29034}}{7x-9999}$$

Answer: The highest power of x in the denominator is clearly x. In the numerator, the factor x^2 is under the cube root, so we consider the leading term to be

$$\sqrt[3]{x^2} = x^{2/3}.$$

Therefore, looking at the entire expression, the highest power of x is just x. Therefore, dividing numerator and denominator by x, we see that

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt[3]{x^2-3x+29034}}{7x-9999} &= \lim_{x\to\infty} \frac{\frac{1}{x}\sqrt[3]{x^2-3x+29034}}{\frac{1}{x}(7x-9999)} \\
&= \lim_{x\to\infty} \frac{\sqrt[3]{\frac{1}{x^3}\left(x^2-3x+29034\right)}}{7-\frac{9999}{x}} \\
&= \lim_{x\to\infty} \frac{\sqrt[3]{\frac{1}{x}-\frac{3}{x^2}+\frac{29034}{x^3}}}{7-\frac{9999}{x}} \\
&= 0.
\end{aligned}$$

5. Let

$$f(x) = \begin{cases} cx^2 - 3 & \text{if } x \leq 2 \\ cx + 2 & \text{if } x > 2 \end{cases}$$

f is continuous provided c equals what value?

Answer: Since both $cx^2 - 3$ and $cx + 2$ are polynomials, they're continuous everywhere, meaning that $f(x)$ is continuous everywhere except possibly at $x = 2$. In order for f to be continuous at 2, it must be the case that $f(2) = \lim_{x \to 2} f(x)$. Now,

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \left(cx^2 - 3\right) = c(2)^2 - 3 = 4c - 3,$$

which is also the value of $f(2)$. On the other hand,

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (cx + 2) = c(2) + 2 = 2c + 2.$$

f will be continuous when these two one-sided limits are equal, meaning when

$$4c - 3 = 2c + 2.$$

Solving for c, we see that f is continuous when

$$c = \frac{5}{2}.$$

6. Is the function f defined below continuous? If not, where is it discontinuous?

$$f(x) = \begin{cases} \sqrt{-x} & \text{if } x < 0 \\ 3 - x & \text{if } 0 \leq x < 3 \\ (3 - x)^2 & \text{if } x \geq 3 \end{cases}$$

Answer: Since each of the three pieces of f is continuous, the only possible discontinuities of f occur where it switches from one piece to another, namely at $x = 0$ and $x = 3$. At $x = 3$,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (3 - x) = 0$$

and

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (3 - x)^2 = 0,$$

so the right-handed and left-handed limit agree. Therefore,

$$\lim_{x \to 3} f(x) = 0,$$

which is equal to $f(3)$, so we conclude that f is continuous at $x = 3$.

On the other hand,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \sqrt{-x} = 0,$$

whereas

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (3 - x) = 3,$$

so f is discontinuous at $x = 0$.

7. Let $f(x)$ be continuous on the closed interval $[-3, 6]$. If $f(-3) = -1$ and $f(6) = 3$, then which of the following must be true?

(a) $f(0) = 0$

(b) $-1 \leq f(x) \leq 3$ for all x between -3 and 6.

(c) $f(c) = 1$ for at least one c between -3 and 6.

(d) $f(c) = 0$ for at least one c between -1 and 3.

Answer: The only one of these statements which is necessarily true is (c): since 1 is between $f(-3) = -1$ and $f(6) = 3$, the Intermediate Value Theorem guarantees that there is some c between -3 and 6 such that $f(c) = 1$. It's coming up with examples to see that none of the other possibilities *has* to be true.

8. Find the one-sided limit
$$\lim_{x \to -1^-} \frac{x-1}{x^4-1}$$
Answer: Notice that, as $x \to -1$, the numerator goes to -2, while the denominator goes to zero. Hence, we would expect the limit to be infinite. However, it could be either $-\infty$ or $+\infty$, so we need to check the sign of the denominator.

When $x < -1$, the quantity $x^4 > 1$, so
$$x^4 - 1 > 0.$$
Therefore, in the one-sided limit, the denominator is always positive. Since the numerator goes to -2, which is negative, the one-sided limit
$$\lim_{x \to -1^-} \frac{x-1}{x^4-1} = -\infty.$$

9. Let
$$f(x) = |x-2|.$$
Does $\lim_{h \to 0} \frac{f(2+h)-f(2)}{h}$ exist? If so, what is this limit?

Answer: This limit does not exist. To see this, I will examine the two one-sided limits:
$$\begin{aligned}\lim_{h \to 0^-} \frac{f(2+h)-f(2)}{h} &= \lim_{h \to 0^-} \frac{|(2+h)-2| - |2-2|}{h} \\ &= \lim_{h \to 0^-} \frac{|h| - 0}{h} \\ &= \lim_{h \to 0^-} \frac{|h|}{h} \\ &= \lim_{h \to 0^-} \frac{-h}{h} \\ &= -1\end{aligned}$$
since $|h| = -h$ when $h < 0$.

On the other hand,
$$\begin{aligned}\lim_{h \to 0^+} \frac{f(2+h)-f(2)}{h} &= \lim_{h \to 0^+} \frac{|(2+h)-2| - |2-2|}{h} \\ &= \lim_{h \to 0^+} \frac{|h|}{h} \\ &= \lim_{h \to 0^+} \frac{h}{h} \\ &= 1\end{aligned}$$
since $|h| = h$ when $h > 0$.

Therefore, since the two one-sided limits don't agree, the limit does not exist.

10. Evaluate

$$\lim_{x\to\infty} \frac{\sqrt{4x^2-8x+7}}{17x+12}$$

Answer: The highest power of x that we see is just x (since the x^2 in the numerator is under a square root). Therefore, dividing both numerator and denominator by x yields

$$\begin{aligned}
\lim_{x\to\infty} \frac{\frac{1}{x}\sqrt{4x^2-8x+7}}{\frac{1}{x}(17x+12)} &= \lim_{x\to\infty} \frac{\sqrt{\frac{1}{x^2}(4x^2-8x+7)}}{17+\frac{12}{x}} \\
&= \lim_{x\to\infty} \frac{\sqrt{4-\frac{8}{x}+\frac{7}{x^2}}}{17+\frac{12}{x}} \\
&= \frac{\sqrt{4}}{17} \\
&= \frac{2}{17}.
\end{aligned}$$

11. Let

$$f(x) = \begin{cases} \frac{x-2}{x^2-4} & \text{for } x \neq 2 \\ a & \text{for } x = 2 \end{cases}$$

If $f(x)$ is continuous at $x = 2$, then find the value of a.

Answer: In order for f to be continuous at $x = 2$, we must have that

$$a = f(2) = \lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x-2}{x^2-4}.$$

Now, we can factor the denominator in the limit to get

$$\lim_{x\to 2} \frac{x-2}{x^2-4} = \lim_{x\to 2} \frac{x-2}{(x+2)(x-2)} = \lim_{x\to 2} \frac{1}{x+2} = \frac{1}{4},$$

so we see that, in order for f to be continuous, a must be $\frac{1}{4}$.

12. Are there any solutions to the equation $\cos x = x$?

Answer: Yes, there is a solution to the equation. To see this, let

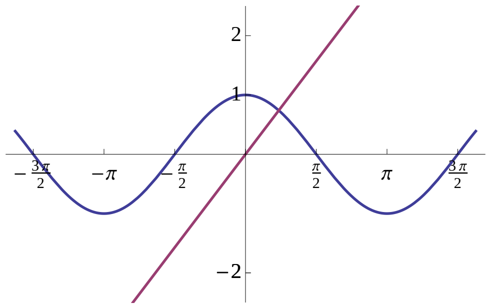
$$f(x) = \cos x - x.$$

Then $f(x) = 0$ precisely when x is a solution to the equation $\cos x = x$, so the problem is to show that $f(x) = 0$ for some x. To see this, notice that f is continuous (since both $\cos x$ and x are continuous functions) and

$$\begin{aligned}
f(0) &= \cos 0 - 0 = 1 > 0 \\
f(\pi) &= \cos \pi - \pi = -1 - \pi < 0.
\end{aligned}$$

Therefore, by the Intermediate Value Theorem there exists c between 0 and π such that $f(c) = 0$. Then, as noted above, $\cos c = c$, so c is a solution to the equation.

Intuitively, we can see that $\cos x = x$ has a solution by looking at the graphs of $y = \cos x$ and $y = x$ (see below); they intersect in exactly one point, so the solution c that we proved exists is actually the only solution.



13. Determine the following limits, if they exist

   (a) $\lim_{x\to -1} \frac{x^2-2x+1}{x-1}$
   Answer: As $x \to -1$ the numerator goes to 4 and the denominator goes to -2, so

$$\lim_{x\to -1} \frac{x^2-2x+1}{x-1} = \frac{4}{-2} = -2.$$

   (b) $\lim_{x\to 1^-} \frac{x^2+2x+1}{x-1}$
   Answer: Notice that

$$\lim_{x\to 1} \left(x^2+2x+1\right) = 1+2+1 = 4,$$

   so the numerator is going to 4. Also, the denominator is going to zero, so we expect the limit to be $\pm\infty$. To see which, notice that, if $x < 1$, then

$$x - 1 < 0,$$

   so the denominator is a very small negative number as $x \to 1^-$. Hence,

$$\lim_{x\to 1^-} \frac{x^2+2x+1}{x-1} = -\infty.$$

14. For each of the following, either evaluate the limit or explain why it doesn't exist.

   (a)

$$\lim_{x\to 16} \frac{4-\sqrt{x}}{x-16}$$

   Answer: I'm going to consider $h(x) = \frac{4-\sqrt{x}}{x-16}$. Rationalizing the denominator yields

$$\begin{aligned}\lim_{x\to 16} \frac{4-\sqrt{x}}{x-16} &= \lim_{x\to 16} \frac{4-\sqrt{x}}{x-16} \cdot \frac{4+\sqrt{x}}{4+\sqrt{x}} \\ &= \lim_{x\to 16} \frac{16-x}{(x-16)(4+\sqrt{x})} \\ &= \lim_{x\to 16} \frac{-(x-16)}{(x-16)(4+\sqrt{x})} \\ &= \lim_{x\to 16} \frac{-1}{4+\sqrt{x}} \\ &= -\frac{1}{8}.\end{aligned}$$

(b)

$$\lim_{x\to-\infty} \frac{6x^2}{\sqrt{7x^4+9}}$$

Answer: Notice that the highest power of x is x^2 (since the x^4 is under a square root). To evaluate this limit, then, I want to multiply numerator and denominator by $\frac{1}{x^2}$:

$$\lim_{x\to-\infty} \frac{\frac{1}{x^2}\left(6x^2\right)}{\frac{1}{x^2}\sqrt{7x^4+9}} = \lim_{x\to-\infty} \frac{6}{\sqrt{\frac{1}{x^4}\left(7x^4+9\right)}} = \lim_{x\to-\infty} \frac{6}{\sqrt{7+\frac{9}{x^4}}}.$$

Since $\frac{9}{x^4}$ goes to zero as $x \to -\infty$, we see that

$$\lim_{x\to-\infty} \frac{6}{\sqrt{7+\frac{9}{x^4}}} = \frac{6}{\sqrt{7}}.$$

Therefore, we can conclude that

$$\lim_{x\to-\infty} \frac{6x^2}{\sqrt{7x^4+9}} = \frac{6}{\sqrt{7}}.$$

(c)

$$\lim_{x\to-2} \frac{x^2+x-2}{x^2+4x+4}$$

Answer: I claim that this limit does not exist. To see why, notice, first of all, that

$$\frac{x^2+x-2}{x^2+4x+4} = \frac{(x+2)(x-1)}{(x+2)(x+2)} = \frac{x-1}{x+2}$$

so long as $x \neq -2$. Therefore, if it exists, $\lim_{x\to-2} \frac{x^2+x-2}{x^2+4x+4}$ must be equal to

$$\lim_{x\to-2} \frac{x-1}{x+2}.$$

Notice that, as $x \to -2$, the numerator goes to -3, while the denominator goes to zero. However, the sign of the denominator depends on which direction x approaches -2 from. When x approaches -2 from the left, we have that

$$\lim_{x\to-2^-} \frac{x-1}{x+2} = +\infty,$$

since both the numerator and denominator are negative. However, as x approaches -2 from the right, we have that

$$\lim_{x\to-2^+} \frac{x-1}{x+2} = -\infty,$$

since the numerator is negative and the denominator is positive.

Therefore, since the two one-sided limits do not agree, the given limit does not exist.

15. (a) At which numbers is the function $h(x) = \cos\left(\frac{x}{1-x^2}\right)$ continuous? Justify your answer.

Answer: I claim that $h(x)$ is continuous whenever $x \neq \pm 1$. To see this, notice that $g(x) = \frac{x}{1-x^2}$ is a rational function, so it is continuous wherever it is defined. Since this function is defined so long as $1 - x^2 \neq 0$, we see that it is defined for all $x \neq \pm 1$.

In turn, the function $f(x) = \cos x$ is continuous, and we know that the composition of continuous functions is continuous. Hence,

$$h(x) = (f \circ g)(x)$$

is continuous wherever it is defined, namely for all $x \neq \pm 1$.

(b) What is $\lim_{x\to 0} h(x)$? Explain your reasoning.

Answer: From part (a), we know that $h(x)$ is continuous at $x = 0$. Therefore, by definition of continuity,

$$\lim_{x\to 0} h(x) = h(0) = \cos\left(\frac{0}{1-0^2}\right) = \cos(0) = 1.$$