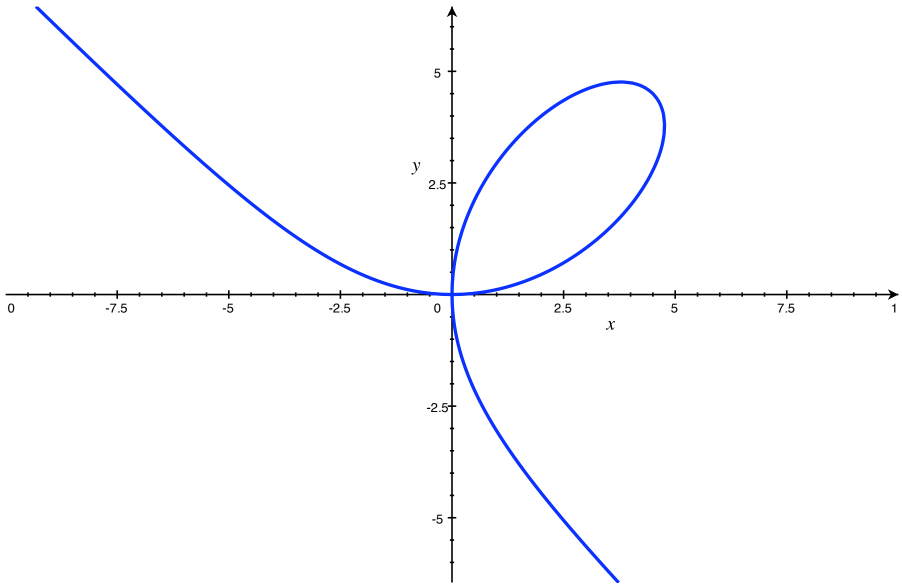
Math 2250 Written HW #8 Solutions

1. The folium of Descartes, pictured below, is determined by the equation

$$x^3 + y^3 - 9xy = 0.$$



(a) Determine the slopes of the tangent lines to the folium at the points $(4, 2)$ and $(2, 4)$.

Answer: We intend to determine y' at these points using implicit differentiation, so differentiate both sides of $x^3 + y^3 - 9xy = 0$ with respect to x:

$$\begin{aligned}\frac{d}{dx}\left(x^3 + y^3 - 9xy\right) &= \frac{d}{dx}(0)\\ 3x^2 + 3y^2y' - 9(1 \cdot y + xy') &= 0\\ 3x^2 + 3y^2y' - 9y - 9xy' &= 0\\ 3x^2 - 9y + y'(3y^2 - 9x) &= 0.\end{aligned}$$

Therefore, we have

$$y'(3y^2 - 9x) = 9y - 3x^2$$

or, equivalently,

$$y' = \frac{9y - 3x^2}{3y^2 - 9x}.$$

Hence, the slope of the tangent line at $(4, 2)$ is

$$y' = \frac{9(2) - 3(4)^2}{3(2)^2 - 9(4)} = \frac{18 - 48}{12 - 36} = \frac{-30}{-24} = \frac{5}{4}.$$

Likewise, the slope of the tangent line at $(2, 4)$ is

$$y' = \frac{9(4) - 3(2)^2}{3(4)^2 - 9(2)} = \frac{36 - 12}{48 - 18} = \frac{24}{36} = \frac{4}{5}.$$

(b) At what point other than the origin does the folium have a horizontal tangent line?

Answer: The folium will have a horizontal tangent line when $y' = 0$. Since we know $y' = \frac{9y-3x^2}{3y^2-9x}$, this will occur exactly when

$$9y - 3x^2 = 0.$$

Obviously, there are infinitely many such points, but we only care about the ones that are actually on the folium, which is to say, those that *also* satisfy the equation $x^3+y^3-9xy = 0$. If a point (x, y) satisfies both of these equations then, from the first, we know that

$$y = \frac{x^2}{3}.$$

Plugging this in to the second, then, we see that

$$x^3 + \left(\frac{x^2}{3}\right)^3 - 9x\left(\frac{x^2}{3}\right) = 0$$

or, equivalently,

$$x^3 + \frac{x^6}{27} - 3x^3 = 0.$$

Combining and factoring, this means

$$x^3\left(\frac{x^3}{27} - 2\right) = 0,$$

so either $x = 0$ or $\frac{x^3}{27} = 2$, meaning $x = \sqrt[3]{54} = 3\sqrt[3]{2}$.

$x = 0$ just gives the origin, so we're interested in the point where $x = 3\sqrt[3]{2}$. The corresponding y-coordinate is

$$y = \frac{(3\sqrt[3]{2})^2}{3} = \frac{9\sqrt[3]{4}}{3} = 3\sqrt[3]{4}.$$

Therefore, the point other than the origin where the folium has a horizontal tangent line is $\left(3\sqrt[3]{2}, 3\sqrt[3]{4}\right)$.

(c) At what point other than the origin does the folium have a vertical tangent line?

Answer: Since we know $y' = \frac{9y-3x^2}{3y^2-9x}$, the curve will have a vertical tangent line when the denominator is zero, meaning

$$3y^2 - 9x = 0.$$

Again, we only care about such points that are also on the folium, meaning they also satisfy the equation $x^3 + y^3 - 9xy = 0$.

But notice that we arrive at these equations from the equations in (b) above simply by interchanging x and y. Therefore, the answer to this part will just be the answer to (b) with the x- and y-coordinates interchanged. In other words, the folium has a vertical tangent at the point

$$\left(3\sqrt[3]{4}, 3\sqrt[3]{2}\right).$$

2. Let $f(x) = x^{\cos x}$. What is $f'(x)$?

   Answer: I will use logarithmic differention. First, take the natural log of both sides:

$$\ln(f(x)) = \ln(x^{\cos x}),$$

so

$$\ln(f(x)) = \cos x \ln x.$$

Now, differentiate both sides with respect to x:

$$\begin{aligned}\frac{d}{dx}(\ln(f(x))) &= \frac{d}{dx}(\cos x \ln x)\\ \frac{1}{f(x)}f'(x) &= -\sin x \ln x + \cos x \frac{1}{x}\\ \frac{f'(x)}{f(x)} &= \frac{\cos x}{x} - \sin x \ln x.\end{aligned}$$

Therefore,

$$f'(x) = f(x)\left(\frac{\cos x}{x} - \sin x \ln x\right).$$

Substituting in $f(x) = x^{\cos x}$, we see that

$$f'(x) = x^{\cos x}\left(\frac{\cos x}{x} - \sin x \ln x\right).$$

3. The length ℓ of a rectangle is decreasing at a rate of 5 cm/sec while the width w is increasing at a rate of 3 cm/sec. At the moment when $\ell = 11$ cm and $w = 9$ cm, determine the following rates of change:

   (a) The rate of change of the area (in cm^2/sec).

   Answer: First, let's get a handle on what we know. We know that the area is given by

$$A(t) = \ell(t)w(t).$$

Also, we know $\ell'(t) = -5$, $w'(t) = 3$, $\ell(t_0) = 11$, and $w(t_0) = 9$.
In this part, we're trying to determine $A'(t_0)$, so we just need to differentiate our expression for $A(t)$ and evaluate at $t = t_0$. Differentiating $A(t) = \ell(t)w(t)$ yields

$$A'(t) = \ell'(t)w(t) + \ell(t)w'(t).$$

Therefore,

$$A'(t_0) = \ell'(t_0)w(t_0) + \ell(t_0)w'(t_0) = (-5)(9) + (11)(3) = -45 + 33 = -12.$$

Hence, we can see that the area is decreasing at a rate of 12 cm^2/sec at this moment.

(b) The rate of change of the perimeter (in cm/sec).

Answer: Now, we're interested in the perimeter of the rectangle, which is given by

$$P(t) = 2\ell(t) + 2w(t).$$

Hence,

$$P'(t) = 2\ell'(t) + 2w'(t),$$

so we have

$$P'(t_0) = 2\ell'(t_0) + 2w'(t_0) = 2(-5) + 2(3) = -10 + 6 = -4.$$

At this moment, then, the perimeter is decreasing at a rate of 4 cm/sec.

(c) The rate of change of the diagonals (in cm/sec).

Answer: From the Pythagorean theorem, if g is the length of a diagonal,

$$g(t)^2 = \ell(t)^2 + w(t)^2.$$

We want to determine $g'(t_0)$, so let's differentiate the above expression:

$$2g(t)g'(t) = 2\ell(t)\ell'(t) + 2w(t)w'(t),$$

so we have

$$g'(t) = \frac{\ell(t)\ell'(t) + w(t)w'(t)}{g(t)}.$$

Now, at time $t = t_0$, we know that

$$g(t_0)^2 = \ell(t_0)^2 + w(t_0)^2 = 11^2 + 9^2 = 121 + 81 = 202,$$

so $g(t_0) = \sqrt{202}$.

Therefore,

$$g'(t_0) = \frac{(11)(-5) + (9)(3)}{\sqrt{202}} = \frac{-55 + 27}{\sqrt{202}} = \frac{-28}{\sqrt{202}};$$

hence the diagonals are decreasing in length at $\frac{28}{\sqrt{202}}$ cm/sec.