Math 2250 Written HW #6 Solutions

1. Let
$$s = \frac{t}{2t+1}$$
Using the *definition* of the derivative (i.e. no quotient rule, no power rule), compute $\frac{ds}{dt}$.

Answer: By definition,
$$\begin{aligned}
\frac{ds}{dt} &= \lim_{h\to 0} \frac{s(t+h)-s(t)}{h} \\
&= \lim_{h\to 0} \frac{\frac{t+h}{2(t+h)+1} - \frac{t}{2t+1}}{h} \\
&= \lim_{h\to 0} \frac{\frac{t+h}{2t+2h+1} - \frac{t}{2t+1}}{h} \\
&= \lim_{h\to 0} \frac{\frac{(2t+1)(t+h)}{(2t+1)(2t+2h+1)} - \frac{t(2t+2h+1)}{(2t+1)(2t+2h+1)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{2t^2+t+2th+h}{(2t+1)(2t+2h+1)} - \frac{2t^2+2th+t}{(2t+1)(2t+2h+1)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{h}{(2t+1)(2t+2h+1)}}{h} \\
&= \lim_{h\to 0} \frac{1}{(2t+1)(2t+2h+1)} \\
&= \frac{1}{(2t+1)^2}.
\end{aligned}$$

2. Consider the curve $y = x^3 - 4x + 1$.

   (a) Find an equation for the line perpendicular to the curve (i.e., perpendicular to the tangent line to the curve) at the point $(2, 1)$. (Recall that perpendicular lines have negative reciprocal slopes.)

   Answer: The line perpendicular to the curve at $(2, 1)$ will have slope equal to the negative reciprocal of the slope of the tangent line. Therefore, we should first determine the slope of the tangent line, which is given by the derivative of the function at the point. For any x,
$$y'(x) = 3x^2 - 4.$$
   Therefore, the derivative at the point $(2, 1)$ is
$$y'(2) = 3(2)^2 - 4 = 8.$$
   Thus, the slope of the tangent line to the curve at $(2, 1)$ is equal to 8, meaning that the slope of the perpendicular line will be equal to $-1/8$. In other words, the perpendicular line is the line of slope $-1/8$ which goes through the point $(2, 1)$. By the point-slope formula, then, this is the line
$$y - 1 = -\frac{1}{8}(x-2),$$

or, equivalently,

$$y = -\frac{x}{8} + \frac{5}{4}.$$

(b) Considering all points on the curve, which has the tangent line with the smallest slope?

Answer: As we saw above, for any point (x, y) on the curve, the slope of the tangent line at that point is equal to

$$y'(x) = 3x^2 - 4.$$

Therefore, the x value of the point with the smallest slope will be the x which makes the above expression as small as possible. Since $3x^2$ is always non-negative, $3x^2 - 4$ has a minimum value of -4, which is achieved when $x = 0$.

Hence, the point on the original curve $y = x^3 - 4x + 1$ with x value 0 will be the point with the smallest slope; this is clearly the point $(0, 1)$.

(c) Find the equations for the tangent lines to the curve at all points where the slope of the tangent line is 8.

Answer: Again, we know that the slope of the tangent line at any point (x, y) on the curve is given by

$$y'(x) = 3x^2 - 4.$$

Therefore, a point (x_0, y_0) on the curve has a tangent line with slope 8 if and only if

$$3x_0^2 - 4 = 8.$$

This happens when $3x_0^2 = 12$, meaning $x_0^2 = 4$, so the tangent line has slope 8 when $x_0 = \pm 2$.

When $x_0 = 2$, then $y_0 = x_0^3 - 4x_0 + 1 = 2^3 - 4(2) + 1 = 1$, so the point $(2, 1)$ has a tangent line with slope 8. Therefore, by the point-slope formula, the equation of the tangent line at this point is

$$y - 1 = 8(x - 2),$$

or, equivalently,

$$y = 8x - 15.$$

Similarly, when $x_0 = -2$, then $y_0 = x_0^3 - 4x_0 + 1 = (-2)^3 - 4(-2) + 1 = 1$, so the point $(-2, 1)$ also has a tangent line with slope 8. By the point-slope formula, this tangent line is given by the equation

$$y - 1 = 8(x - (-2))$$

or

$$y = 8x + 17.$$