Math 2250 Written HW #5 Solutions

1. What is the rate of change of the volume of a ball (remember that the volume of a ball of radius r is given by $\frac{4}{3}\pi r^3$) with respect to the radius when the radius is $r = 2$?

   Answer: Think of the volume V of the ball as a function of r. Specifically,

   $$V(r) = \frac{4}{3}\pi r^3.$$

   Therefore, the rate of change of the volume of the ball with respect to the radius when $r = 2$ is given by $V'(2)$. Using the definition of the derivative,

   $$\begin{aligned} V'(2) &= \lim_{h\to 0} \frac{V(2+h) - V(2)}{h} \\ &= \lim_{h\to 0} \frac{\left[\frac{4}{3}\pi(2+h)^3\right] - \left[\frac{4}{3}\pi(2)^3\right]}{h} \\ &= \lim_{h\to 0} \frac{\frac{4}{3}\pi\left(8 + 12h + 6h^2 + h^3\right) - \frac{4}{3}\pi \cdot 8}{h} \\ &= \lim_{h\to 0} \frac{16\pi h + 8\pi h^2 + \frac{4}{3}\pi h^3}{h} \\ &= \lim_{h\to 0} \left[16\pi + 8\pi h + \frac{4}{3}\pi h^2\right] \\ &= 16\pi. \end{aligned}$$

   Hence, the desired rate of change is 16π.

2. Show that the line $y = mx + b$ is its own tangent line at any point (x_0, y_0) on the line.

   Answer: Let $f(x) = mx + b$. Pick a point (x_0, y_0) on the line $y = mx + b$. Since the point is on the line, it must be the case that $y_0 = mx_0 + b$.

   Now, the slope of the tangent line to $y = mx + b$ at the point (x_0, y_0) is given by $f'(x_0)$. By definition,

   $$\begin{aligned} f'(x_0) &= \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} \\ &= \lim_{h\to 0} \frac{[m(x_0+h) + b] - [mx_0 + b]}{h} \\ &= \lim_{h\to 0} \frac{(mx_0 + mh + b) - (mx_0 + b)}{h} \\ &= \lim_{h\to 0} \frac{mh}{h} \\ &= \lim_{h\to 0} m \\ &= m. \end{aligned}$$

   Therefore, by the point-slope formula, the tangent line is given by

   $$y - y_0 = m(x - x_0).$$

Using the fact that $y_0 = mx_0 + b$, this becomes

$$y - mx_0 - b = mx - mx_0.$$

Adding $mx_0 + b$ to both sides tells us that the tangent line is

$$y = mx + b,$$

as desired.