Math 2250 Written HW #3 Solutions

1. At what values of x is the function $f(x) = \frac{x \tan x}{x^2+1}$ continuous?

   Answer: It's most convenient to think of $f(x)$ as the product of the functions $g(x) = \frac{x}{x^2+1}$ and $h(x) = \tan x$:

   $$f(x) = \frac{x \tan x}{x^2 + 1} = \frac{x}{x^2 + 1} \tan x = g(x)h(x).$$

   Now, the function $g(x)$ has continuous numerator and denominator, and the denominator is never zero (since $x^2 + 1 \geq 1$ for all x), so, by the Limit Laws, $g(x)$ is continuous everywhere.

   On the other hand, $h(x) = \tan x = \frac{\sin x}{\cos x}$. Both $\sin x$ and $\cos x$ are continuous everywhere, so $h(x)$ will be continuous wherever the denominator is not zero. The function $\cos x$ is equal to zero for

   $$x = \ldots, -3\pi/2, -\pi/2, \pi/2, 3\pi/2, 5\pi/2, \ldots.$$

   In other words, $\cos x$ equals zero whenever $x = \frac{(2k+1)\pi}{2}$ for any integer k.

   Therefore, by the Limit Laws, the function $h(x)$ is continuous whenever x is *not* equal to $\frac{(2k+1)\pi}{2}$.

   Finally, since $g(x)$ is continuous everywhere, $h(x)$ is continuous except when $x = \frac{(2k+1)\pi}{2}$, and $f(x) = g(x) \cdot h(x)$, we can conclude that $f(x)$ is continuous for all x not equal to $\frac{(2k+1)\pi}{2}$.

2. For what value of a is the function

   $$f(x) = \begin{cases} x^2 - 1 & \text{if } x < 5 \\ 2ax & \text{if } x \geq 5 \end{cases}$$

   continuous at every x?

   Answer: Notice that for all $x < 5$, $f(x) = x^2 - 1$. Since $x^2 - 1$ is a polynomial and polynomials are continuous, we can see that $f(x)$ is continuous on $(-\infty, 5)$.

   Likewise, for all $x > 5$, $f(x) = 2ax$. The functions $2ax$ is a polynomial, and hence continuous, for any choice of a, so we know that $f(x)$ is continuous on $(5, +\infty)$.

   Therefore, the only *possible* point of discontinuity for f is at $x = 5$.

   In order for $f(x)$ to be continuous at $x = 5$, we must have that

   $$\lim_{x \to 5} f(x) = f(5) \tag{1}$$

   (this is the definition of continuity).

   In particular, the left hand side of the above equation must exist. In turn, that means that we need

   $$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) \tag{2}$$

   to hold. Let's compute each side separately.

   Dealing with the left hand side first, we know that for all $x < 5$, we have $f(x) = x^2 - 1$. Hence,

   $$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} (x^2 - 1) = 5^2 - 1 = 24.$$

On the other hand, to compute the right hand side of equation (2), note that for all $x > 5$, $f(x) = 2ax$. Therefore,

$$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} 2ax = 2a \cdot 5 = 10a.$$

Combining the above, then, we see that equation (2) is true if and only if

$$24 = 10a$$

or, equivalently,

$$a = \frac{24}{10} = \frac{12}{5}.$$

Remember that equation (2) was what we needed to be true in order for $\lim_{x \to 5} f(x)$ to exist. Hence, the limit exists when $a = 12/5$, so from here on out we will assume $a = 12/5$.

Now we can turn to verifying (1). Since $\lim_{x \to 5} f(x)$ exists, it must be equal to both $\lim_{x \to 5^-} f(x)$ and $\lim_{x \to 5^+} f(x)$; in particular,

$$\lim_{x \to 5} f(x) = \lim_{x \to 5^-} f(x) = 24$$

as computed above.

On the other hand, the right hand side of (1) is given by

$$f(5) = 2a \cdot 5 = 10a = 10 \cdot \frac{12}{5} = 24.$$

Since the left and right hand sides are equal, equation (1) is valid, so we can conclude that $f(x)$ is continuous at $x = 5$. Since we already knew that $f(x)$ was continuous at all other values of x, we know that $f(x)$ is continuous at all x when $a = \frac{12}{5}$.

3. Use the Intermediate Value Theorem to show that the equation

$$x^3 - 15x + 1 = 0$$

has (at least) three solutions on the interval $[-4, 4]$.

Answer: Let $f(x) = x^3 - 15x + 1$. Notice, first of all, that $f(x)$ is a polynomial, and hence continuous everywhere. Therefore, we can apply the Intermediate Value Theorem on any interval we please. Now, I compute several values of the function:

$$\begin{aligned} f(0) &= 0^3 - 15(0) + 1 = 0 - 0 + 1 = 1 \\ f(1) &= 1^3 - 15(1) + 1 = 1 - 15 + 1 = -13 \\ f(4) &= 4^3 - 15(4) + 1 = 64 - 60 + 1 = 5 \\ f(-4) &= (-4)^3 - 15(-4) + 1 = -64 + 60 + 1 = -3. \end{aligned}$$

Notice that $f(-4) < 0$ and $f(0) > 0$, so the Intermediate Value Theorem guarantees that there exists a number c_1 between -4 and 0 so that $f(c_1) = 0$.

Likewise, $f(0) > 0$ and $f(1) < 0$, so the Intermediate Value Theorem guarantees that there exists a number c_2 between 0 and 1 so that $f(c_2) = 0$.

Finally, $f(1) < 0$ and $f(4) > 0$, so the Intermediate Value Theorem guarantees that there exists a number c_3 between 1 and 4 so that $f(c_3) = 0$.

Hence, the given equation holds for $x = c_1, c_2$, and c_3. Each of these numbers is between -4 and 4, so we conclude that the equation has at least three solutions on the interval $[-4, 4]$.