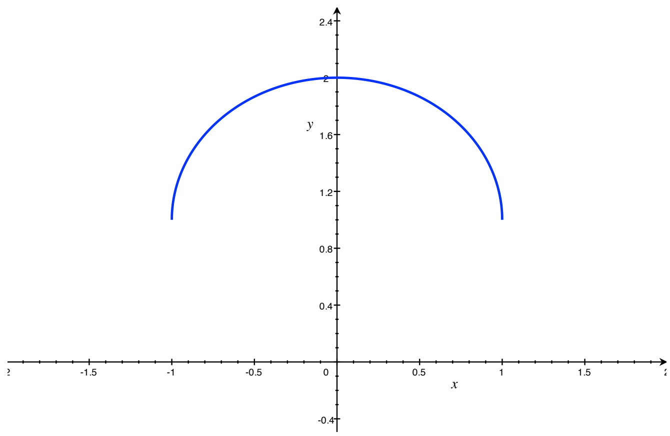
Math 2250 Written HW #15 Solutions

1. Graph the function $f(x) = 1 + \sqrt{1-x^2}$ and use some geometry (and *not* the Fundamental Theorem of Calculus) to compute the definite integral

$$\int_{-1}^{1} \left(1 + \sqrt{1-x^2}\right) dx.$$

Answer: Recall that $y = \sqrt{1-x^2}$ is the equation of a semicircle of radius 1 centered at the origin and lying above the x-axis. Adding 1 shifts the graph up by 1, so the graph of $y = 1 + \sqrt{1-x^2}$ looks like this:



Hence, the area under the curve is equal to the area of a rectangle of width 2 and height 1 plus the area of a semicircle of radius 1, which is $\frac{1}{2}\pi(1)^2 = \pi/2$. Therefore,

$$\int_{-1}^{1} \left(1 + \sqrt{1-x^2}\right) dx = 2 + \frac{\pi}{2}.$$

2. Using properties of the definite integral, show that if $f(x) \geq 0$ for all x in the interval $[a, b]$, then

$$\int_a^b f(x)dx \geq 0.$$

In other words, the definite integral of a non-negative function is non-negative.

Answer: Using property 6 from the list I wrote on the board in class, since $f(x) \geq 0$ for all x in $[a, b]$, we know that

$$\int_a^b f(x) \geq 0 \cdot (b-a) = 0,$$

as desired.

3. Compute the definite integral

$$\int_0^{\pi} \frac{1}{2} \left(\cos x + |\cos x|\right) dx.$$

Answer: For x between 0 and $\pi/2$, $\cos x$ is nonnegative, so $|\cos x| = \cos x$. On the other hand, $\cos x \leq 0$ for x between $\pi/2$ and π, so $|\cos x| = -\cos x$ on $[\pi/2, \pi]$. Therefore, using property 5 of the definite integral,

$$\begin{aligned}\int_0^\pi \frac{1}{2}(\cos x + |\cos x|)\,dx &= \int_0^{\pi/2} \frac{1}{2}(\cos x + |\cos x|)\,dx + \int_{\pi/2}^\pi \frac{1}{2}(\cos x + |\cos x|)\,dx \\ &= \int_0^{\pi/2} \frac{1}{2}(\cos x + \cos x)\,dx + \int_{\pi/2}^\pi \frac{1}{2}(\cos x - \cos x)\,dx \\ &= \int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^\pi 0\,dx.\end{aligned}$$

The second term is zero, so we just need to compute $\int_0^{\pi/2} \cos x\,dx$. Using the Fundamental Theorem of Calculus, since $\sin x$ is an antiderivative of $\cos x$, we know that

$$\int_0^{\pi/2} \cos x\,dx = [\sin x]_0^{\pi/2} = \sin(\pi/2) - \sin(0) = 1 - 0 = 1.$$

Putting this all together, then, we see that $\int_0^\pi \frac{1}{2}(\cos x + |\cos x|)\,dx = 1$.