Math 2250 Written HW #13 Solutions

1. Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3. An illustration is given on p. 261, problem #12. Remember that the volume of a circular cone with radius r and height h is $\frac{1}{3}\pi r^2 h$. (*Hint: which variable you choose to express the volume in terms of makes a big difference in how easy it is to differentiate*)

   Answer: First, notice that, by the Pythagorean Theorem,

   $$x^2 + y^2 = 3^2,$$

   meaning that

   $$x^2 = 9 - y^2.$$

   Also, since the volume of a cone with radius r and height h is $\frac{1}{3}\pi r^2 h$, we know that the volume of the cone is

   $$\frac{1}{3}\pi x^2(3+y) = \frac{1}{3}\pi(9-y^2)(3+y) = \frac{1}{3}\pi\left[27 + 9y - 3y^2 - y^3\right].$$

   Therefore, we want to maximize the function $V(y) = \frac{1}{3}\pi\left[27 + 9y - 3y^2 - y^3\right]$ subject to the constraint $0 \leq y \leq 3$.

   To find the critical points, we differentiate:

   $$V'(y) = \frac{1}{3}\pi\left[9 - 6y - 3y^2\right] = \pi\left[3 - 2y - y^2\right] = \pi(3+y)(1-y).$$

   Therefore, $V'(y) = 0$ when

   $$\pi(3+y)(1-y) = 0,$$

   meaning that $y = -3$ or $y = 1$. Only $y = 1$ is in the interval $[0, 3]$, so that's the only critical point we need to concern ourselves with.

   Now we evaluate V at the critical point and the endpoints:

   $$\begin{aligned} V(0) &= \frac{1}{3}\pi\left[27 + 9(0) - 3(0)^2 - 0^3\right] = 9\pi \\ V(1) &= \frac{1}{3}\pi\left[27 + 9(1) - 3(1)^2 - 1^3\right] = \frac{32\pi}{3} \\ V(3) &= \frac{1}{3}\pi\left[27 + 9(3) - 3(3)^2 - 3^3\right] = 0. \end{aligned}$$

   Therefore, the volume of the largest cone that can be inscribed in a sphere of radius 3 is $\frac{32\pi}{3}$.

2. Problem #24 from p. 262 of the textbook.

   Answer: The volume of the trough is simply the area of the cross-section times the length, which is 20.

   Now, to determine the area of the cross-section, notice that, if b is the base of one of the right triangles, then

   $$\sin\theta = \frac{b}{1} = b.$$

Likewise, if h is the height of one of the right triangles, then

$$\cos\theta = \frac{h}{1} = h.$$

Therefore, the area of each right triangle is

$$\frac{1}{2}bh = \frac{1}{2}\sin\theta\cos\theta.$$

In turn, the area of the rectangle is

$$1 \cdot h = 1 \cdot \cos\theta = \cos\theta.$$

Therefore, the area of the cross-section is the sum of the area of the rectangle and the areas of the two triangles:

$$A = \cos\theta + \frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\sin\theta\cos\theta = \cos\theta + \sin\theta\cos\theta.$$

Therefore, the volume of the trough is

$$V(\theta) = 20A = 20(\cos\theta + \sin\theta\cos\theta) = 20\cos\theta + 20\sin\theta\cos\theta,$$

which we want to maximize subject to the constraint $0 \leq \theta \leq \pi/2$.

First, differentiate to find critical points:

$$V'(\theta) = -20\sin\theta + 20\cos^2\theta - 20\sin^2\theta.$$

Notice that $\cos^2\theta - \sin^2\theta = \cos 2\theta$, so we can simplify $V'(\theta)$ as

$$V'(\theta) = -20\sin\theta + 20\cos 2\theta.$$

Therefore, $V'(\theta) = 0$ when

$$-20\sin\theta + 20\cos 2\theta = 0,$$

or, equivalently, when

$$\cos 2\theta = \sin\theta.$$

On the interval $[0, \pi/2]$, this only happens at $\theta = \pi/6$, so this is our critical point.

Now, check the critical point and the endpoints:

$$\begin{aligned}
V(0) &= 20\cos(0) + 20\sin(0)\cos(0) = 20 \\
V(\pi/6) &= 20\cos(\pi/6) + 20\sin(\pi/6)\cos(\pi/6) = 20 \cdot \frac{\sqrt{3}}{2} + 20 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = 15\sqrt{3} \\
V(\pi/2) &= 20\cos(\pi/2) + 20\sin(\pi/2)\cos(\pi/2) = 0.
\end{aligned}$$

Since $15\sqrt{3} > 20$, we see that the maximum volume occurs when the angle $\theta = \pi/6$.