Math 2250 Written HW #12 Solutions

1. Consider

$$f(x) = \frac{1}{4x^2 + 7}$$

(a) Find all critical points of $f(x)$. For each one determine whether it is a local maximum, a local minimum, or neither.

Answer: To find the critical points, first we need to compute

$$f'(x) = \frac{-1}{(4x^2+7)^2} \cdot 8x = \frac{-8x}{(4x^2+7)^2}.$$

Since the denominator is always positive, $f'(x) = 0$ exactly when the numerator is zero, so the only critical point occurs when $x = 0$, so this is the point $(0, 1/7)$.

Now, again since the denominator is always positive, $f'(x)$ has the same sign as the numerator, which is $-8x$. Therefore, $f'(x)$ changes sign from positive to negative at $x = 0$, so the first derivative test implies that $(0, 1/7)$ is a local maximum.

(b) If it exists, what is the absolute maximum of $f(x)$? If it exists, what is the absolute minimum?

Answer: Since $f(x)$ has only one critical point, and that critical point is a local maximum, it must also be an absolute maximum. Therefore, the absolute maximum value of $f(x)$ is $1/7$. On the other hand, any absolute minimum would also have to be a local minimum; since we've seen that there are no local minima, we can conclude that $f(x)$ does not have an absolute minimum.

(c) Find the inflection points of $f(x)$ and determine on what intervals the graph of $f(x)$ is concave up and on what intervals it is concave down.

Answer: To determine the inflection points, we first need to determine $f''(x)$, which we do by differentiating $f'(x)$ using the quotient rule:

$$\begin{aligned} f''(x) &= \frac{(4x^2+7)^2(-8) - (-8x)(2(4x^2+7)(8x))}{(4x^2+7)^4} \\ &= \frac{8(4x^2+7)\left[16x^2 - (4x^2+7)\right]}{(4x^2+7)^4} \\ &= \frac{8(12x^2-7)}{(4x^2+7)^3} \end{aligned}$$

Therefore, $f''(x) = 0$ when $12x^2 - 7 = 0$, meaning when $x = \pm\sqrt{\frac{7}{12}}$.

Notice that for $x < -\sqrt{\frac{7}{12}}$ or $x > \sqrt{\frac{7}{12}}$, we have that $12x^2 - 7 > 0$, so $f''(x)$ is positive and $f(x)$ is concave up, whereas for $-\sqrt{\frac{7}{12}} < x < \sqrt{\frac{7}{12}}$ we have $12x^2 - 7 < 0$, so $f''(x)$ is negative and $f(x)$ is concave down.

Therefore, the second derivative changes sign at both $x = -\sqrt{\frac{7}{12}}$ and at $x = \sqrt{\frac{7}{12}}$, so these are both inflection points, and $f(x)$ is concave up on the intervals

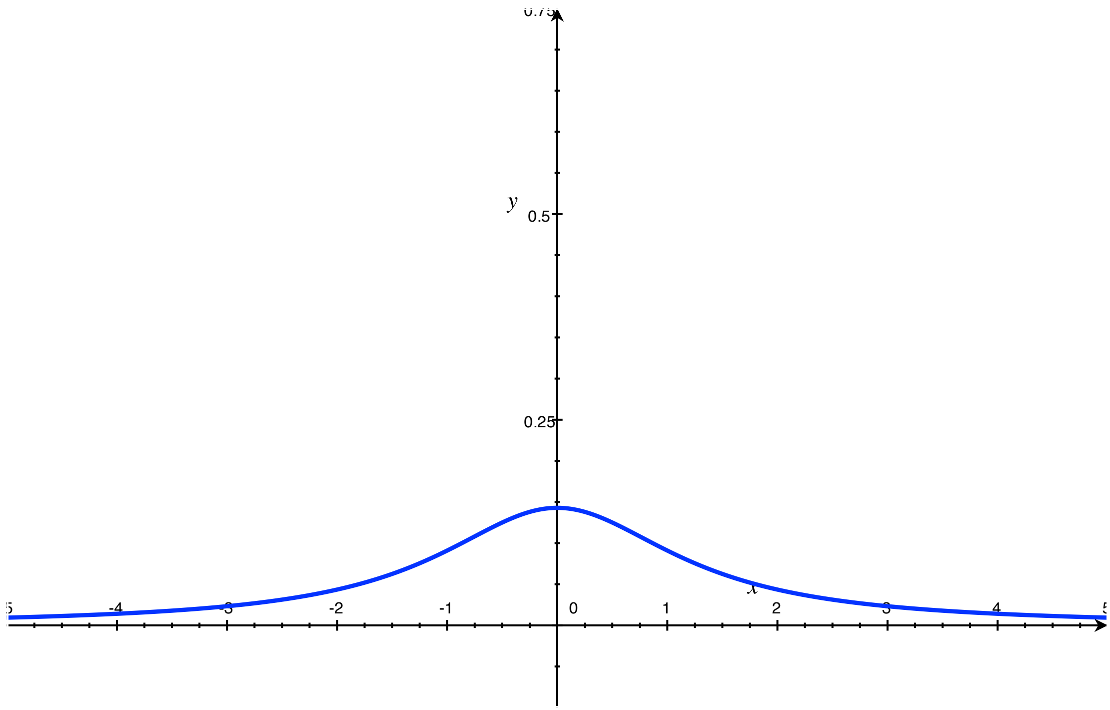
$$\left(-\infty, -\sqrt{\frac{7}{12}}\right) \quad \text{and} \quad \left(\sqrt{\frac{7}{12}}, +\infty\right)$$

and concave down on the interval

$$\left(-\sqrt{\frac{7}{12}}, \sqrt{\frac{7}{12}}\right).$$

(d) Using the information in (a)–(c), sketch the graph of $f(x)$.

Answer:



2. Evaluate the limit

$$\lim_{x\to 0} \frac{8x(1-\cos x)}{x - \sin x}$$

Answer: Notice that $\lim_{x\to 0}(8x(1-\cos x)) = 0$ and $\lim_{x\to 0}(x - \sin x) = 0$, so we can apply L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{8x(1-\cos x)}{x-\sin x} = \lim_{x\to 0} \frac{\frac{d}{dx}\left[8x(1-\cos x)\right]}{\frac{d}{dx}[x-\sin x]} = \lim_{x\to 0} \frac{8x\sin x + 8(1-\cos x)}{1-\cos x} = \lim_{x\to 0} \frac{8 - 8\cos x + 8x\sin x}{1-\cos x}.$$

Again both numerator and denominator are going to zero, so we can apply L'Hôpital's Rule again to get

$$\lim_{x\to 0} \frac{\frac{d}{dx}\left[8 - 8\cos x + 8x\sin x\right]}{\frac{d}{dx}\left[1-\cos x\right]} = \lim_{x\to 0} \frac{8\sin x + 8\sin x + 8x\cos x}{\sin x} = \lim_{x\to 0} \frac{16\sin x + 8x\cos x}{\sin x}.$$

Once *again* the numerator and denominator are both going to zero, so we apply L'Hôpital's Rule one more time to get

$$\lim_{x\to 0} \frac{\frac{d}{dx}\left[16\sin x + 8x\cos x\right]}{\frac{d}{dx}[\sin x]} = \lim_{x\to 0} \frac{16\cos x + 8\cos x - 8x\sin x}{\cos x} = \lim_{x\to 0} \frac{24\cos x - 8x\sin x}{\cos x}.$$

But now both numerator and denominator are continuous, so we can just evaluate at $x = 0$ to see that the above limit is equal to $\frac{24(1)-8(0)(0)}{1} = 24$.

Therefore, we conclude that the original limit is

$$\lim_{x\to 0} \frac{8x(1-\cos x)}{x - \sin x} = 24.$$

3. Evaluate the limit

$$\lim_{x\to -\infty} x^3 e^x.$$

Answer: Since $\lim_{x\to-\infty} x^3 = -\infty$ and $\lim_{x\to-\infty} e^x = 0$, we have a $\infty \cdot 0$ situation. To get it in a form suitable for using L'Hôpital's Rule, we re-write the above limit as

$$\lim_{x\to -\infty} \frac{x^3}{1/e^x} = \lim_{x\to -\infty} \frac{x^3}{e^{-x}}.$$

The numerator is going to $-\infty$ and the denominator is going to $+\infty$ so we can apply L'Hôpital's Rule to get

$$\lim_{x\to -\infty} \frac{\frac{d}{dx}[x^3]}{\frac{d}{dx}[e^{-x}]} = \lim_{x\to -\infty} \frac{3x^2}{-e^{-x}}.$$

Now the numerator is going to $+\infty$ and the denominator is going to $-\infty$, so we can use L'Hôpital's Rule again to get

$$\lim_{x\to -\infty} \frac{\frac{d}{dx}[3x^2]}{\frac{d}{dx}[-e^{-x}]} = \lim_{x\to -\infty} \frac{6x}{e^{-x}}.$$

Apply L'Hôpital's Rule one more time to get

$$\lim_{x\to -\infty} \frac{\frac{d}{dx}[6x]}{\frac{d}{dx}[e^{-x}]} = \lim_{x\to -\infty} \frac{6}{-e^{-x}}.$$

But now the numerator is constant and the denominator is going to $-\infty$, so we see that this limit is equal to zero. Therefore, we conclude that

$$\lim_{x\to -\infty} x^3 e^x = 0.$$