Math 2250 Written HW #11 Solutions

1. Show that the function $f(x) = x^4 + 3x + 1$ has exactly one zero in the interval $[-2, -1]$. (*Hint: to show that f has at least one zero, use the Intermediate Value Theorem as in Section 2.5*)

   Answer: We attack this in two parts: (i) show that $f(x)$ has at least one zero; and (ii) show that $f(x)$ has at most one zero.

   For (i), as suggested in the hint, the goal is to use the Intermediate Value Theorem. The IVT is surely applicable since $f(x)$ is a polynomial and, hence, continuous everywhere. Now,

$$f(-2) = (-2)^4 + 3(-2) + 1 = 16 - 6 + 1 = 11 > 0$$
$$f(-1) = (-1)^4 + 3(-1) + 1 = 1 - 3 + 1 = -2 < 0.$$

   Therefore, by the Intermediate Value Theorem there exists some number c between -2 and -1 so that $f(c) = 0$, and so we see that $f(x)$ has at least one zero in the interval $[-2, -1]$.

   For (ii), the strategy is to use the Mean Value Theorem. First, notice that

$$f'(x) = 4x^3 + 3.$$

   Since $4x^3 + 3$ is negative for all x in $[-2, -1]$, we see that $f'(x) < 0$ for all x in $[-2, -1]$.

   On the other hand, if there were *more* than one zero of $f(x)$ in this interval, there would be at least two, and so we could pick two, say a and b with $a < b$. Then the Mean Value Theorem would imply that there exists c in (a, b) so that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{0 - 0}{b - a} = 0.$$

   But we just saw that this is impossible since $f'(x) < 0$ for all x in $[-2, -1]$.

   Therefore, we can conclude that there is at most one zero of $f(x)$ in $[-2, -1]$. Combined with (i), then, this implies that there is exactly one zero of $f(x)$ in $[-2, -1]$.

2. Suppose the acceleration of an oscillating particle is given by

$$a(t) = -4\sin(2t)$$

   at that the particle's position at time $t = 0$ is -3 and its velocity at time $t = 0$ is 2. Find the particle's position as a function of t.

   Answer: Since $a(t) = v'(t)$, if we can find some function $g(t)$ so that $g'(t) = a(t) = v'(t)$, then we'll know that $v(t) = g(t) + C$ for some constant C, which we can then solve for using $v(0) = 2$.

   To find such a $g(t)$, notice $\cos(2t)$ is the *sort* of function that has a derivative more or less like $-4\sin(2t)$. Specifically,

$$\frac{d}{dt}(\cos(2t)) = -\sin(2t) \cdot 2 = -2\sin(2t).$$

   Therefore, to get a function that has $a(t)$ as a derivative, we should multiply $\cos(2t)$ by 2:

$$\frac{d}{dt}(2\cos(2t)) = -2\sin(2t) \cdot 2 = -4\sin(2t).$$

Thus,

$$v(t) = 2\cos(2t) + C$$

for some constant C which we now solve for using $v(0) = 2$:

$$\begin{aligned} v(0) &= 2\cos(2 \cdot 0) + C \\ 2 &= 2\cos(0) + C \\ 2 &= 2 + C \end{aligned}$$

so $C = 0$, and we have that $v(t) = 2\cos(2t)$.

Next, we use the same strategy to determine the position function $s(t)$, using the fact that $s'(t) = v(t)$. So now we seek a function with $2\cos(2t)$ as its derivative. This is a bit easier:

$$\frac{d}{dt}(\sin(2t)) = \cos(2t) \cdot 2 = 2\cos(2t).$$

Hence,

$$s(t) = \sin(2t) + D$$

for some constant D which we can determine using $s(0) = -3$:

$$\begin{aligned} s(0) &= \sin(2 \cdot 0) + D \\ -3 &= 0 + D \\ -3 &= D. \end{aligned}$$

From all this, then, we see that the position of the particle at time t is given by the function

$$s(t) = \sin(2t) - 3.$$

3. A trucker handed in a ticket at a toll booth showing that in 2 hours he had covered 159 miles on a toll road with a speed limit of 65 mph. He was immediately cited for speeding and, when he asked for an explanation, the only response was "Mean Value Theorem, son." Explain.

   Answer: Let $s(t)$ denote the position of the trucker at time t (starting at $t = 0$ when he enters the toll road), and let $v(t) = s'(t)$ be his velocity. Then, by the Mean Value Theorem, there exists some time t_0 so that

$$v(t_0) = s'(t_0) = \frac{s(2) - s(0)}{2 - 0} = \frac{159}{2} = 79.5.$$

   That means there was some time when the trucker was driving exactly 79.5 mph, which is clearly well above the 65 mph speed limit; hence the speeding ticket.