Math 215 HW #2 Solutions

1. Problem 1.4.6. Write down the 2 by 2 matrices A and B that have entries $a_{ij} = i + j$ and $b_{ij} = (-1)^{i+j}$. Multiply them to find AB and BA.

   Solution: Since a_{ij} indicates the entry in A which is in the ith row and in the jth column, we see that
$$A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}.$$
   Likewise,
$$B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$
   Therefore,
$$AB = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 + 3 \cdot (-1) & 2 \cdot (-1) + 3 \cdot 1 \\ 3 \cdot 1 + 4 \cdot (-1) & 3 \cdot (-1) + 4 \cdot 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}.$$
   Also,
$$BA = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 + (-1) \cdot 3 & 1 \cdot 3 + (-1) \cdot 4 \\ (-1) \cdot 2 + 1 \cdot 3 & (-1) \cdot 3 + 1 \cdot 4 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}.$$

2. Problem 1.4.16. Let x be the column vector $(1, 0, \ldots, 0)$. Show that the rule $(AB)x = A(Bx)$ forces the first column of AB to equal A times the first column of B.

   Solution: Suppose that x has n components. Then, in order for Bx to make sense, B must be an $m \times n$ matrix for some m. In turn, for the matrix product AB to make sense, A must be an $\ell \times m$ matrix for some ℓ.

   Now, suppose
$$A = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{\ell 1} & \cdots & a_{\ell m} \end{bmatrix}$$
   and
$$B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix}.$$
   Then
$$AB = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{\ell 1} & \cdots & a_{\ell m} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^m a_{1i}b_{i1}] & \cdots & \sum_{i=1}^m a_{1i}b_{in} \\ \vdots & & \vdots \\ \sum_{i=1}^m a_{\ell i}b_{i1} & \cdots & \sum_{i=1}^m a_{\ell i}b_{in} \end{bmatrix}.$$
   Hence,
$$(AB)x = \begin{bmatrix} \sum_{i=1}^m a_{1i}b_{i1}] & \cdots & \sum_{i=1}^m a_{1i}b_{in} \\ \vdots & & \vdots \\ \sum_{i=1}^m a_{\ell i}b_{i1} & \cdots & \sum_{i=1}^m a_{\ell i}b_{in} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^m a_{1i}b_{i1} \\ \sum_{i=1}^m a_{2i}b_{i1} \\ \vdots \\ \sum_{i=1}^m a_{\ell i}b_{i1} \end{bmatrix},$$

which is just a copy of the first column of AB.

On the other hand,

$$Bx = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{m1} \end{bmatrix},$$

which is the first column of B. Therefore, $A(Bx)$ is A times the first column of B; since $A(Bx) = (AB)x$ and $(AB)x$ is the first column of AB, we see that the first column of AB must be A times the first column of B.

Though it's not part of the assigned problem, the same argument with different choices of x (e.g. $(0, 1, 0, \ldots, 0)$, etc.) will demonstrate that each column of AB must be equal to A times the corresponding column in B.

3. Problem 1.4.20. The matrix that rotates the xy-plane by an angle θ is

$$A(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Verify that $A(\theta_1)A(\theta_2) = A(\theta_1 + \theta_2)$ from the identities for $\cos(\theta_1 + \theta_2)$ and $\sin(\theta_1 + \theta_2)$. What is $A(\theta)$ times $A(-\theta)$?

Solution: Using the definition of $A(\theta_1)$ and $A(\theta_2)$, we have that

$$\begin{aligned} A(\theta_1)A(\theta_2) &= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & -\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2 \\ \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 & -\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix} \\ &= A(\theta_1+\theta_2), \end{aligned}$$

where I went from the second to the third lines using the identities for $\cos(\theta_1 + \theta_2)$ and $\sin(\theta_1 + \theta_2)$. Geometrically, the fact that $A(\theta_1)A(\theta_2) = A(\theta_1 + \theta_1)$ corresponds to the fact that rotating something by an angle of θ_2 and then rotating the result by an angle of θ_1 is the same as rotating the original object by an angle of $\theta_1 + \theta_2$.

Now, letting $\theta_1 = \theta$ and $\theta_2 = -\theta$, the above implies that

$$A(\theta)A(-\theta) = A(\theta + (-\theta)) = A(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This corresponds to intuition: rotating something clockwise through some angle and then rotating counterclockwise through the same angle has the same end result as not moving the object at all.

4. Problem 1.4.32. Write these ancient problems in a 2 by 2 matrix form $Ax = b$ and solve them:

(a) X is twice as old as Y and their ages add to 39.

Solution: We can interpret this problem as the following system of equations:

$$\begin{aligned} X &= 2Y \\ X + Y &= 39 \end{aligned}$$

or, equivalently,

$$\begin{aligned} X - 2Y &= 0 \\ X + Y &= 39. \end{aligned}$$

Define the matrix

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix};$$

then it's clear that the following matrix equation of the form $Ax = b$ is equivalent to the given problem:

$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 \\ 39 \end{bmatrix}.$$

Subtracting the first row from the second yields the equation

$$\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 \\ 39 \end{bmatrix},$$

which we can now solve by back-substitution. Clearly, $3Y = 39$, so $Y = 13$. Hence,

$$0 = X - 2Y = X - 26,$$

so $X = 26$. Thus, X is 26 years old and Y is 13 years old.

(b) $(x, y) = (2, 5)$ and $(3, 7)$ lie on the line $y = mx + c$. Find m and c.

Solution: Since the two given points lie on the line, we have that

$$\begin{aligned} 2m + c &= 5 \\ 3m + c &= 7. \end{aligned}$$

Therefore, we can re-interpret the problem in terms of solving the following matrix equation:

$$\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}.$$

Subtracting $\frac{3}{2}$ times the first row from the second row yields

$$\begin{bmatrix} 2 & 1 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} 5 \\ -\frac{1}{2} \end{bmatrix}.$$

Hence,

$$-\frac{1}{2}c = -\frac{1}{2},$$

meaning that $c = 1$. In turn, we have that

$$5 = 2m + c = 2m + 1,$$

so $m = 2$. Therefore, the given points lie on the line

$$y = 2x + 1.$$

5. Problem 1.4.56. What 2 by 2 matrix P_1 projects the vector (x, y) onto the x-axis to produce $(x, 0)$? What matrix P_2 projects onto the y-axis to produce $(0, y)$? If you multiply $(5, 7)$ by P_1 and then multiply by P_2, you get (____) and (____).

   Solution: Just to give the components of P_1 names, suppose

$$P_1 = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}.$$

   Then P_1 times (x, y) is given by

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p_{11}x + p_{12}y \\ p_{21}x + p_{22}y \end{bmatrix}.$$

   Since this is supposed to equal $(x, 0)$, we see that

$$\begin{aligned} p_{11}x + p_{12}y &= x \\ p_{21}x + p_{22}y &= 0. \end{aligned}$$

   Therefore, since the above equations must hold for any possible values of x and y, it must be the case that $p_{11} = 1$, $p_{12} = 0$, $p_{21} = 0$, and $p_{22} = 0$. Therefore,

$$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

   On the other hand, suppose

$$P_2 = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}.$$

   Then P_2 times (x, y) is

$$\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} q_{11}x + q_{12}y \\ q_{21}x + q_{22}y \end{bmatrix}.$$

   Since this is supposed to equal $(0, y)$, we see that

$$\begin{aligned} q_{11}x + q_{12}y &= 0 \\ q_{21}x + q_{22}y &= y. \end{aligned}$$

   Again, since these equations must hold for any possible choices of x and y, we see that $q_{11} = q_{12} = q_{21} = 0$ and $q_{22} = 1$. Therefore,

$$P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now, multiply $(5,7)$ by P_1:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}.$$

Multiplying this, in turn, by P_2 yields

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This, of course, makes perfect sense, as projecting $(5,7)$ sequentially to the x-axis (multiplying by P_1) and then to the y-axis (multiplying by P_2) will of course yield the zero vector.

6. Problem 1.5.4. Apply elimination to produce the factors L and U for

$$A = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 8 \end{bmatrix}.$$

Solution: For the first A, we get U by subtracting 4 times row 1 from row 2:

$$U = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

and the row operation is recorded by the matrix L:

$$L = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}.$$

For the second A, the first steps in elimination are to subtract $1/3$ of row 1 from rows 2 and 3, yielding

$$\begin{bmatrix} 3 & 1 & 1 \\ 0 & \frac{8}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & \frac{8}{3} \end{bmatrix}.$$

The last step, then, is to subtract $1/4$ of row 2 from row 3, yielding

$$U = \begin{bmatrix} 3 & 1 & 1 \\ 0 & \frac{8}{3} & \frac{2}{3} \\ 0 & 0 & \frac{15}{6} \end{bmatrix}.$$

These three row operations are recorded in the matrix L:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & 1 \end{bmatrix}.$$

For the third A, the first steps in elimination are to subtract the first row from the second and third rows, yielding

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 3 & 7 \end{bmatrix}.$$

Next, subtract the second row from the third to get

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{bmatrix}.$$

The elimination steps are recorded by L:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

7. Problem 1.5.18. Decide whether the following systems are singular or nonsingular, and whether they have no solution, one solution, or infinitely many solutions:

$$\begin{array}{rcrcrcl} & & v & - & w & = & 2 \\ u & - & v & & & = & 2 \\ u & & & - & w & = & 2 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcl} & & v & - & w & = & 0 \\ u & - & v & & & = & 0 \\ u & & & - & w & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcl} & & v & + & w & = & 1 \\ u & + & v & & & = & 1 \\ u & & & + & w & = & 1 \end{array}$$

Solution: Let

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

A is a matrix which records the left sides of the first two systems of equations.

Then, in order to do elimination on A, first switch rows 1 and 2 to get

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}.$$

Next, subtract row 1 from row 3 to get

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Finally, subtract row 2 from row 3 to get

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the bottom row is zero, we see that the matrix A is singular (and, thus, the first two systems of equations are singular). Applying the same row operations to the vector $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$, which represents the right-hand side of the first system, yields

$$\begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}.$$

Therefore, the first system is equivalent to the matrix equation

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix},$$

which implies that $0 = -2$. Since this is clearly impossible, the first system is inconsistent and has no solutions.

On the other hand, applying the above row operations to $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ doesn't change it, so the second system is equivalent to the matrix equation

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

which has infinitely many solutions.

Turning to the third system of equations, let

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Then the first elimination step is to switch the first and second rows, yielding

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}. \tag{$*$}$$

This row switch is recorded by the permutation matrix

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Getting back to ($*$), the next elimination step is to subtract row 1 from row 3:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Finally, adding row 2 to row 3 yields

$$U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Note that this implies that the original matrix A (and, thus, the third system of equations) was nonsingular, so we should expect a unique solution.

These row operations are recorded by the matrix

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}.$$

Then, with A, P, U, and L as above,

$$PA = LU.$$

Now, to solve the corresponding system of equations, note that the right hand side is unaffected by the permutation matrix P. Hence, $Lc = Pb$ is just the equation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Hence, $c_1 = 1$ and $c_2 = 1$. In turn, the bottom row yields

$$1 = c_1 - c_2 + c_3 = 1 - 1 + c_3,$$

so $c_3 = 1$.

Thus, solving $Ux = c$ for x means solving

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

We see that $2w = 1$, so $w = 1/2$. In turn,

$$1 = v + w = v + 1/2,$$

so $v = 1/2$. Finally,

$$1 = u + v = u + 1/2,$$

so $u = 1/2$. Therefore, the unique solution to the third system of equations is

$$(u, v, w) = (1/2, 1/2, 1/2).$$

8. Problem 1.5.24. What three elimination matrices E_{21}, E_{31}, E_{32} put A into upper triangular form $E_{32}E_{31}E_{21}A = U$? Multiply by E_{32}^{-1}, E_{31}^{-1}, and E_{21}^{-1} to factor A into LU where $L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$. Find L and U:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}.$$

Solution: The first two elimination steps are to subtract 2 times row 1 from row 2 and to subtract 3 times row 1 from row 3, yielding:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & -1 \end{bmatrix}. \qquad (\dagger)$$

These two steps correspond to the matrices

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

Getting back to (†), the final elimination step is to subtract twice row 2 from row 3, yielding

$$U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

This elimination step corresponds to the matrix

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix},$$

and it's easy to check that, with A, U, E_{21}, E_{31}, and E_{32} as described,

$$E_{32}E_{31}E_{21}A = U.$$

The action of E_{21} is undone by adding twice row 1 to row 2, so

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Likewise, the action of E_{31} is undone by adding three times row 2 to row 3 and the action of E_{32} is undone by adding twice row 2 to row 3, so

$$E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}.$$

Thus,

$$L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}.$$

9. Problem 1.5.30. Find L and U for the nonsymmetric matrix

$$A = \begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix}.$$

Find the four conditions on a, b, c, d, r, s, t to get $A = LU$ with four pivots.

Solution: Provided $a \neq 0$, the first elimination steps are to subtract row 1 from rows 2, 3, and 4:

$$\begin{bmatrix} a & r & r & r \\ 0 & b-r & s-r & s-r \\ 0 & b-r & c-r & t-r \\ 0 & b-r & c-r & d-r \end{bmatrix}.$$

Provided $b - r \neq 0$, we have a good pivot in the second row and we can subtract row 2 from rows 3 and 4:

$$\begin{bmatrix} a & r & r & r \\ 0 & b-r & s-r & s-r \\ 0 & 0 & (c-r)-(s-r) & (t-r)-(s-r) \\ 0 & 0 & (c-r)-(s-r) & (d-r)-(s-r) \end{bmatrix} = \begin{bmatrix} a & r & r & r \\ 0 & b-r & s-r & s-r \\ 0 & 0 & c-s & t-s \\ 0 & 0 & c-s & d-s \end{bmatrix}.$$

Provided $c - s \neq 0$, we have a good pivot in the third row and we can subtract row 3 from row 4 to get

$$U = \begin{bmatrix} a & r & r & r \\ 0 & b-r & s-r & s-r \\ 0 & 0 & c-s & t-s \\ 0 & 0 & 0 & (d-s)-(t-s) \end{bmatrix} = \begin{bmatrix} a & r & r & r \\ 0 & b-r & s-r & s-r \\ 0 & 0 & c-s & t-s \\ 0 & 0 & 0 & d-t \end{bmatrix}.$$

So long as $d - t \neq 0$, this is a nonsingular matrix. The above row operations are encoded in the matrix

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Putting all this together, we see that $A = LU$ with L and U as above under the conditions

$$a \neq 0, \quad b \neq r, \quad c \neq s, \quad d \neq t.$$

10. Problem 1.5.42. If P_1 and P_2 are permutation matrices, so is P_1P_2. This still has the rows of I in some order. Give examples with $P_1P_2 \neq P_2P_1$ and $P_3P_4 = P_4P_3$.

    Solution: Define P_1 and P_2 by

$$P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

    In other words, P_1 switches the first and second rows, while P_2 switches the second and third rows.

    Then

$$P_1P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

On the other hand

$$P_2P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Hence, $P_1P_2 \neq P_2P_1$. (Notice: we could have figured this out directly without actually performing the matrix multiplication. P_1P_2 corresponds to applying P_2, then applying P_1. Given that P_2 switches rows 2 and 3 and that P_1 switches rows 1 and 2, applying P_2 and then P_1 corresponds to sending the first row to the second row, the second row to the third row, and the third row to the first row. On the other hand, P_2P_1 corresponds to applying P_1, then P_2, which sends the first row to the third row, the third row to the second row, and the second row to the first row.)

Now, let P_3 and P_4 be given by

$$P_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad P_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

In other words, P_3 switches the first two rows and P_4 switches the last two rows. It's clear that the order that we apply P_3 and P_4 shouldn't matter: switching rows 1 and 2 first and then switching rows 3 and 4 yields the same final result as switching rows 3 and 4 first and then switching rows 1 and 2. We can confirm this by performing the matrix multiplication:

$$P_3P_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$P_4P_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

so, indeed, $P_3P_4 = P_4P_3$.

(Note: the identity matrix is, technically, a permutation matrix, so the easy way out would have been to choose, say, $P_3 = I$; since the identity matrix commutes with everything, any choice of P_4 would have yielded $P_3P_4 = P_4P_3$.)