Math 215 HW #1 Solutions

1. Problem 1.2.3. Describe the intersection of the three planes $u+v+w+z = 6$ and $u+w+z = 4$ and $u + w = 2$ (all in four-dimensional space). Is it a line or a point or an empty set? What is the intersection if the fourth plane $u = -1$ is included? Find a fourth equation that leaves us with no solution.

   Solution: We can plug the third equation, $u+w = 2$, into the second equation, $u+w+z = 4$, to see that
$$z = 2.$$
   In turn, plugging $u + w + z = 4$ into the first equation, $u + v + w + z = 6$, yields
$$w = 2.$$
   Hence, points (u, v, w, z) in the intersection of the three planes are described by
$$\begin{aligned} u + w &= 2 \\ v &= 2 \\ z &= 2. \end{aligned}$$
   The latter two equations specify a plane parallel to the uw-plane (but with $v = z = 2$ instead of $v = z = 0$). Within this plane, the equation $u + w = 2$ describes a line (just as it does in the uw-plane), so we see that the three planes intersect in a line.

   Adding the fourth equation $u = -1$ shrinks the intersection to a point: plugging $u = -1$ into $u + w = 2$ gives that $w = 3$. Hence, the intersection is at the single point
$$(u, v, w, z) = (-1, 2, 3, 2).$$
   Any fourth equation which is inconsistent with the first three will leave us with no solution. For example, $v + z = 5$ (this is inconsistent with the first three equations since those three specify that $v = z = 2$, meaning that $v + z = 4$).

2. Problem 1.2.4. Sketch these three lines and decide if the equations are solvable:
$$\begin{aligned} x + 2y &= 2 \\ x - y &= 2 \\ y &= 1. \end{aligned}$$
   What happens if all right-hand sides are zero? Is there any nonzero choice of right-hand sides that allows the three lines to intersect at the same point?

   Solution: See Figure 1. This system of equations is not solvable, as the three lines specified by the three equations do not intersect in a common point.

   If the right hand sides are all changed to zero, then $(x, y) = (0, 0)$ is certainly a solution to the system.

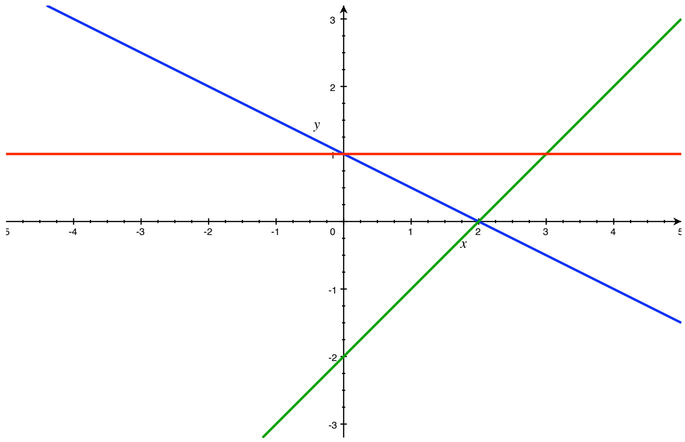

Figure 1:

Consider the new system

$$\begin{aligned} x + 2y &= 6 \\ x - y &= 3 \\ y &= 1. \end{aligned}$$

Then it's easy to check that $(x, y) = (4, 1)$ is a solution of this system of equations.

3. Problem 1.2.8. Explain why the system

$$\begin{aligned} u + v + w &= 2 \\ u + 2v + 3w &= 1 \\ v + 2w &= 0 \end{aligned}$$

is singular by finding a combination of the three equations that adds up to $0 = 1$. What value should replace the last zero on the right side to allow the equations to have solutions—and what is one of the solutions?

Solution: Subtract the second line from the first, then add the third line. This yields $0 = 1$. Hence, these three equations cannot simultaneously be true (since, if they were, it would be true that $0 = 1$).

To find the appropriate replacement for 0 on the right-hand side of the last equation, solve the following system by elimination:

$$\begin{aligned} u + v + w &= 2 \\ u + 2v + 3w &= 1 \\ v + 2w &= a. \end{aligned}$$

Replacing the second row with the second row minus the first yields

$$\begin{aligned} u + v + w &= 2 \\ v + 2w &= -1 \\ v + 2w &= a. \end{aligned}$$

In turn, replacing the third row with the third row minus the second row yields

$$\begin{aligned} u + v + w &= 2 \\ v + 2w &= -1 \\ 0 &= a + 1. \end{aligned}$$

In order for the third row of this new system to be true, it must be the case that $a = -1$. If so, then the second row implies that $v = -1 - 2w$, so the first row becomes

$$u + (-1 - 2w) + w = 2,$$

or, equivalently,

$$u - w = 3.$$

The two equations $u - w = 3$ and $v + 2w = -1$ specify a line of solutions; to find one solution, just let $w = 0$ and solve for u and v. This yields the solution $(u, v, w) = (3, -1, 0)$.

4. Problem 1.2.10. Under what condition on y_1, y_2, y_3 do the points $(0, y_1)$, $(1, y_2)$, $(2, y_3)$ lie on a straight line?

   Solution: The points $(0, y_1)$, $(1, y_2)$, and $(2, y_3)$ will lie on the same line if and only if the slope of the line segment from $(0, y_1)$ to $(1, y_2)$ is the same as the slope of the line segment from $(1, y_2)$ to $(2, y_3)$.

   The slope of the first line segment is

   $$\frac{y_2 - y_1}{1 - 0} = y_2 - y_1$$

   and the slope of the second is

   $$\frac{y_3 - y_2}{2 - 1} = y_3 - y_2,$$

   so the condition is that $y_3 - y_2 = y_2 - y_1$. In other words, the three points will be collinear if and only if

   $$y_3 = 2y_2 - y_1.$$

5. Problem 1.2.22. If (a, b) is a multiple of (c, d) with $abcd \neq 0$, *show that* (a, c) *is a multiple of* (b, d). This is surprisingly important: call it a challenge question. You could use numbers first to see how a, b, c, d are related. The question will lead to:

   $$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ has dependent rows then it has dependent columns.}$$

   Proof. If (a, b) is a multiple of (c, d), then there is some $r \in \mathbb{R}$ such that

   $$(a, b) = r(c, d) = (rc, rd).$$

   Hence,

   $$a = rc = r\frac{c}{d}d = \frac{c}{d}(rd) = \frac{c}{d}b.$$

   Hence,

   $$(a, c) = \left(\frac{c}{d}b, \frac{c}{d}d\right) = \frac{c}{d}(b, d),$$

   so (a, c) is a multiple of (b, d). $\square$

6. Problem 1.3.6. Choose a coefficient b that makes this system singular. Then choose a right-hand side g that makes it solvable. Find two solutions in that singular case.

$$\begin{aligned} 2x + by &= 16 \\ 4x + 8y &= g. \end{aligned}$$

Solution: Using elimination to solve the system, we can replace the second row by the second row minus twice the first to get the new system

$$\begin{aligned} 2x + by &= 16 \\ (8 - 2b)y &= g - 32. \end{aligned}$$

This system is singular, then, if the left hand side of the second row is zero, meaning $8-2b = 0$ or $b = 4$. However, the system will still be solvable if the right hand side of the second row is also zero, meaning $g - 32 = 0$ or $g = 32$.

If we set $b = 4$ and $g = 32$, then the above elimination process tells us that

$$2x + 4y = 16.$$

We can easily find one solution by setting $x = 0$ and another by setting $y = 0$. These two solutions are, respectively, $(x, y) = (0, 4)$ and $(x, y) = (8, 0)$.

7. Problem 1.3.12. Which number d forces a row exchange, and what is the triangular system (not singular) for that d? Which d makes this system singular (no third pivot)?

$$\begin{aligned} 2x + 5y + z &= 0 \\ 4x + dy + z &= 2 \\ y - z &= 3. \end{aligned}$$

Solution: The first step in the elimination procedure is to replace the second row by the second row minus twice the first row. This yields:

$$\begin{aligned} 2x + 5y + z &= 0 \\ (d - 10)y - z &= 2 \\ y - z &= 3, \end{aligned} \tag{*}$$

We will have to exchange the second and third rows if $d - 10 = 0$, meaning $d = 10$. Provided $d = 10$, then, after this row exchange, the system would be

$$\begin{aligned} 2x + 5y + z &= 0 \\ y - z &= 3 \\ -z &= 2, \end{aligned}$$

which is already a triangular system.

Going back to the system (*), if $d \neq 10$, then the next step of the elimination procedure will be to replace the third row by the third row minus $\frac{1}{d-10}$ times the second, yielding:

$$\begin{aligned}
2x + 5y + z &= 0 \\
(d-10)y - z &= 2 \\
\left(-1 + \frac{1}{d-10}\right) z &= 3 - \frac{2}{d-10}
\end{aligned}$$

This system will be singular if the coefficient on z in the third equation is zero; that is, if

$$\frac{1}{d-10} = 1.$$

Equivalently, the system will be singular if $d = 11$.

8. Problem 1.3.30. Use elimination to solve

$$\begin{aligned}
u + v + w &= 6 \\
u + 2v + 2w &= 11 \\
2u + 3v - 4w &= 3
\end{aligned}$$

and

$$\begin{aligned}
u + v + w &= 7 \\
u + 2v + 2w &= 10 \\
2u + 3v - 4w &= 3.
\end{aligned}$$

Solution: For the first system, we replace the second row by the second row minus the first and we replace the third row by the third row minus twice the second:

$$\begin{aligned}
u + v + w &= 6 \\
v + w &= 5 \\
v - 6w &= -9
\end{aligned}$$

In turn, replacing the third row by the third row minus the second yields

$$\begin{aligned}
u + v + w &= 6 \\
v + w &= 5 \\
-7w &= -14.
\end{aligned}$$

Hence, $w = 2$ and, plugging this into the second equation, $v = 3$. Plugging both of these into the first equation yields $u = 1$, so we see that the unique solution of this system of equations is

$$(u, v, w) = (1, 3, 2).$$

As for the second system of equations, replace the second row by the second row minus the first and replace the third row by the third row minus twice the first to get

$$\begin{aligned} u + v + w &= 7 \\ v + w &= 3 \\ v - 6w &= -11. \end{aligned}$$

Then replace the third row by the third row minus the second:

$$\begin{aligned} u + v + w &= 7 \\ v + w &= 3 \\ -7w &= -14 \end{aligned}$$

(notice that these were the same elimination steps as we did for the first system; the only change is on the right hand side). Then $w = 2$ and, plugging this into the second equation, $v = 1$. Plugging both into the first equation gives $u = 4$. Hence, the unique solution of the system is

$$(u, v, w) = (4, 1, 2).$$

9. Prove that it is impossible for a system of linear equations to have exactly two solutions. Two questions you might think about to get your thinking started: (i) if (x_1, y_1, z_1) and (x_2, y_2, z_2) are two solutions, what is another one? (ii) If 25 planes meet at 2 points, where else do they meet?

Proof. Suppose

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= b_1 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

is a system of linear equations with two distinct solutions, $(y_1, \ldots, y_n)$ and $(z_1, \ldots, z_n)$. I want to show that if there are at least two solutions then there are actually infinitely many solutions. In particular, I will show that every point on the line determined by the two solutions is also a solution.

A point on the line between the two solutions $(y_1, \ldots, y_n)$ and $(z_1, \ldots, z_n)$ is given by

$$(ty_1 + (1-t)z_1, \ldots, ty_n + (1-t)z_n)$$

for some $t \in \mathbb{R}$. I claim that each such point gives another solution to the system. Notice that, if we plug this point into the ith row of the system of equations, we get

$$\begin{aligned} a_{i1}(ty_1 + (1-t)z_1) + \ldots + a_{in}(ty_n + (1-t)z_n) &= t\,(a_{i1}y_1 + \ldots + a_{in}y_n) + (1-t)\,(a_{i1}z_1 + \ldots + a_{in}z_n) \\ &= tb_i + (1-t)b_i \\ &= b_i \end{aligned}$$

since $(y_1, \ldots, y_n)$ and $(z_1, \ldots, z_n)$ are solutions of the system. Therefore, we see that this point satisfies each of the equations in the system, so this point is a solution to the system.

Since the choice of point on the line was arbitrary, we see that every point on the line connecting $(y_1, \ldots, y_n)$ and $(z_1, \ldots, z_n)$ is a solution to the system of equations, so there are infinitely many solutions to the system. $\square$