Math 215 HW #11 Solutions

1. Problem 5.5.6. Find the lengths and the inner product of

$$\vec{x} = \begin{bmatrix} 2-4i \\ 4i \end{bmatrix} \quad \text{and} \quad \vec{y} \begin{bmatrix} 2+4i \\ 4i \end{bmatrix}.$$

Answer: First,

$$\|\vec{x}\|^2 = \vec{x}^H\vec{x} = [2+4i \ \ -4i] \begin{bmatrix} 2-4i \\ 4i \end{bmatrix} = (4+16)+16 = 36,$$

so $\|\vec{x}\| = 6$. Likewise,

$$\|\vec{y}\|^2 = \vec{y}^H\vec{y} = [2-4i \ \ -4i] \begin{bmatrix} 2+4i \\ 4i \end{bmatrix} = (4+16)+16,$$

so $\|\vec{y}\| = 6$.

Finally,

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^H\vec{y} = [2+4i \ \ -4i] \begin{bmatrix} 2+4i \\ 4i \end{bmatrix} = (2+4i)^2 - (4i)^2 = (4-16+16i)+16 = 4+16i.$$

2. Problem 5.5.16. Write one significant fact about the eigenvalues of each of the following:

(a) A real symmetric matrix.

Answer: As we saw in class, the eigenvalues of a real symmetric matrix are all real numbers.

(b) A stable matrix: all solutions to $du/dt = Au$ approach zero.

Answer: By the definition of stability, this means that the reals parts of the eigenvalues of A are non-positive.

(c) An orthogonal matrix.

Answer: If $A\vec{x} = \lambda\vec{x}$, then

$$\langle A\vec{x}, A\vec{x} \rangle = \langle \lambda\vec{x}, \lambda\vec{x} \rangle = \lambda^2 \langle \vec{x}, \vec{x} \rangle = \lambda^2 \|\vec{x}\|^2.$$

On the other hand,

$$\langle A\vec{x}, A\vec{x} \rangle = (A\vec{x})^T A\vec{x} = \vec{x}^T A^T A\vec{x} = \vec{x}^T\vec{x} = \langle \vec{x}, \vec{x} \rangle = \|\vec{x}\|^2.$$

Therefore,

$$\lambda^2 \|\vec{x}\|^2 - \|\vec{x}\|^2,$$

meaning that $\lambda^2 = 1$, so $|\lambda| = 1$.

(d) A Markov matrix.

Answer: We saw in class that $\lambda_1 = 1$ is an eigenvalue of every Markov matrix, and that all eigenvalues λ_i of a Markov matrix satisfy $|\lambda_i| \leq 1$.

(e) A defective matrix (nondiagonalizable).

Answer: If A is $n \times n$ and is not diagonalizable, then A must have fewer than n eigenvalues (if A had n distinct eigenvalues and since eigenvectors corresponding to different eigenvalues are linear independent, then A would have n linearly independent eigenvectors, which would imply that A is diagonalizable).

(f) A singular matrix.

Answer: If A is singular, then A has a non-trivial nullspace, which means that 0 must be an eigenvalue of A.

3. Problem 5.5.22. Every matrix Z can be split into a Hermitian and a skew-Hermitian part, $Z = A + K$, just as a complex number z is split into $a + ib$. The real part of z is half of $z + \overline{z}$, and the "real part" (i.e. Hermitian part) of Z is half of $Z + Z^H$. Find a similar formula for the "imaginary part" (i.e. skew-Hermitian part) K, and split these matrices into $A + K$:

$$Z = \begin{bmatrix} 3+4i & 4+2i \\ 0 & 5 \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} i & i \\ -i & i \end{bmatrix}.$$

Answer: Notice that

$$(Z + Z^H)^H = Z^H + (Z^H)^H = Z^H + Z,$$

so indeed $\frac{1}{2}(Z + Z^H)$ is Hermitian. Likewise,

$$(Z - Z^H)^H = Z^H - (Z^H)^H = Z^H - Z = -(Z - Z^H),$$

is skew-Hermitian, so $K = \frac{1}{2}(Z - Z^H)$ is the skew-Hermitian part of Z.

Hence, when

$$Z = \begin{bmatrix} 3+4i & 4+2i \\ 0 & 5 \end{bmatrix},$$

we have

$$A = \frac{1}{2}(Z+Z^H) = \frac{1}{2}\left(\begin{bmatrix} 3+4i & 4+2i \\ 0 & 5 \end{bmatrix} + \begin{bmatrix} 3-4i & 0 \\ 4-2i & 5 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 6 & 4+2i \\ 4-2i & 10 \end{bmatrix} = \begin{bmatrix} 3 & 4+2i \\ 4-2i & 5 \end{bmatrix}$$

and

$$K = \frac{1}{2}(Z-Z^H) = \frac{1}{2}\left(\begin{bmatrix} 3+4i & 4+2i \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 3-4i & 0 \\ 4-2i & 5 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 8i & 4+2i \\ -4+2i & 0 \end{bmatrix} = \begin{bmatrix} 4i & 4+2i \\ -4+2i & 0 \end{bmatrix}.$$

On the other hand, when

$$Z = \begin{bmatrix} i & i \\ -i & i \end{bmatrix}$$

we have

$$A = \frac{1}{2}(Z + Z^H) = \frac{1}{2}\left(\begin{bmatrix} i & i \\ -i & i \end{bmatrix} + \begin{bmatrix} -i & i \\ -i & -i \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 0 & 2i \\ -2i & 0 \end{bmatrix} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

and

$$K = \frac{1}{2}(Z - Z^H) = \frac{1}{2}\left(\begin{bmatrix} i & i \\ -i & i \end{bmatrix} - \begin{bmatrix} -i & i \\ -i & -i \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 2i & 0 \\ 0 & 2i \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}.$$

4. Problem 5.5.28. If $A\vec{z} = \vec{0}$, then $A^H A\vec{z} = \vec{0}$. If $A^H A\vec{z} = \vec{0}$, multiply by $\vec{z}^H$ to prove that $A\vec{z} = \vec{0}$. The nullspaces of A and A^H are ______. $A^H A$ is an invertible Hermitian matrix when the nullspace of A contains only $\vec{z} =$ ______.

   Answer: Suppose $A^H A\vec{z} = \vec{0}$. Then, multiplying both sides by $\vec{z}^H$ yields

$$0 = \vec{z}^H A^H A\vec{z} = (A\vec{z})^H (A\vec{z}) = \langle A\vec{z}, A\vec{z}\rangle = \|A\vec{z}\|^2,$$

   meaning that $A\vec{z} = \vec{0}$.

   Therefore, we see that if $A\vec{z} = \vec{0}$, then $A^H A\vec{z} = \vec{0}$ and if $A^H A\vec{z} = \vec{0}$, then $A\vec{z} = \vec{0}$, so the nullspaces of A and A^H are equal. $A^H A$ is an invertible matrix only if its nullspace is $\{\vec{0}\}$, so we see that $A^H A$ is an invertible matrix when the nullspace of A contains only $\vec{z} = \vec{0}$.

5. Problem 5.5.48. Prove that the inverse of a Hermitian matrix is again a Hermitian matrix.

   Proof. If A is Hermitian, then

$$A = U\Lambda U^H,$$

   where U is unitary and Λ is a real diagonal matrix. Therefore,

$$A^{-1} = (U\Lambda U^H)^{-1} = (U^H)^{-1}\Lambda^{-1}U^{-1} = U\Lambda^{-1}U^H$$

   since $U^{-1} = U^H$. Note that Λ^{-1} is just the diagonal matrix with entries $1/\lambda_i$ (where the λ_i are the entries in Λ). Hence,

$$(A^{-1})^H = (U\Lambda^{-1}U^H)^H = U(\Lambda^{-1})^H U^H = U\Lambda^{-1}U^H = A^{-1}$$

   since Λ^{-1} is a real matrix, so we see that A^{-1} is Hermitian. □

6. Problem 5.6.8. What matrix M changes the basis $\vec{V}_1 = (1,1)$, $\vec{V}_2 = (1,4)$ to the basis $\vec{v}_1 = (2,5)$, $\vec{v}_2 = (1,4)$? The columns of M come from expressing $\vec{V}_1$ and $\vec{V}_2$ as combinations $\sum m_{ij}\vec{v}_i$ of the $\vec{v}$'s.

   Answer: Since

$$\vec{V}_1 = \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\5\end{bmatrix} - \begin{bmatrix}1\\4\end{bmatrix} = \vec{v}_1 - \vec{v}_2$$

   and

$$\vec{V}_2 = \begin{bmatrix}1\\4\end{bmatrix} = \vec{v}_2,$$

   we see that

$$M = \begin{bmatrix}1 & 0\\-1 & 1\end{bmatrix}.$$

7. Problem 5.6.12. The *identity transformation* takes every vector to itself: $T\vec{x} = \vec{x}$. Find the corresponding matrix, if the first basis is $\vec{v}_1 = (1,2)$, $\vec{v}_2 = (3,4)$ and the second basis is $\vec{w}_1 = (1,0)$, $\vec{w}_2 = (0,1)$. (It is not the identity matrix!)

   Answer: Despite the slightly confusing way this question is worded, it is just asking for the matrix M which converts the $\vec{v}$ basis into the $\vec{w}$ basis. Clearly,

$$\vec{v}_1 = \begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix} = \vec{w}_1 + 2\vec{w}_2$$

and

$$\vec{v}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3\vec{w}_1 + 4\vec{w}_2,$$

so the desired matrix is

$$M = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$

8. Problem 5.6.38. These Jordan matrices have eigenvalues 0, 0, 0, 0. They have two eigenvectors (find them). But the block sizes don't match and J *is not similar to* K.

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For any matrix M, compare JM and MK. If they are equal, show that M is not invertible. Then $M^{-1}JM = K$ is impossible.

Answer: First, we find the eigenvectors of J and K. Since all eigenvalues of both are 0, we're just looking for vectors in the nullspace of J and K. First, for J, we note that J is already in reduced echelon form and that $J\vec{v} = \vec{0}$ implies that $\vec{v}$ is a linear combination of

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Hence, these are the eigenvectors of J.

Likewise, K is already in reduced echelon form and $K\vec{v} = \vec{0}$ implies that $\vec{v}$ is a linear combination of

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Hence, these are the eigenvectors of K.

Now, suppose

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}$$

such that $JM = MK$. Then

$$JM = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} = \begin{bmatrix} m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$MK = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & m_{11} & m_{12} & 0 \\ 0 & m_{21} & m_{22} & 0 \\ 0 & m_{31} & m_{32} & 0 \\ 0 & m_{41} & m_{42} & 0 \end{bmatrix}.$$

Therefore $JM = MK$ means that

$$\begin{bmatrix} m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & m_{11} & m_{12} & 0 \\ 0 & m_{21} & m_{22} & 0 \\ 0 & m_{31} & m_{32} & 0 \\ 0 & m_{41} & m_{42} & 0 \end{bmatrix}$$

and so we have that

$$m_{21} = m_{24} = m_{22} = m_{41} = m_{44} = m_{42}0.$$

Plugging these back into M, we see that

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ 0 & 0 & m_{23} & 0 \\ m_{31} & m_{32} & m_{33} & m_{34} \\ 0 & 0 & m_{43} & 0 \end{bmatrix}.$$

Clearly, the second and fourth rows are multiples of each other, so M cannot possibly have rank 4. However, M not having rank 4 means that M cannot be invertible. Therefore, $M^{-1}JM = K$ is impossible, so it cannot be the case that J and K are similar.

9. Problem 5.6.40. Which pairs are similar? Choose a, b, c, d to prove that the other pairs aren't:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{bmatrix} b & a \\ d & c \end{bmatrix} \quad \begin{bmatrix} c & d \\ a & b \end{bmatrix} \quad \begin{bmatrix} d & c \\ b & a \end{bmatrix}.$$

Answer: The second and third are clearly similar, since

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b & a \\ d & c \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}.$$

Likewise, the first and fourth are similar, since

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} d & c \\ b & a \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c & d \\ a & b \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

There are no other similarities, as we can see by choosing

$$a = 1, \quad b = c = d = 0.$$

Then the matrices are, in order

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Each of these is already a diagonal matrix, and clearly the first and fourth have 1 as an eigenvalue, whereas the second and third have only 0 as an eigenvalue. Since similar matrices have the same eigenvalues, we see that neither the first nor the fourth can be similar to either the second or the third.

10. (**Bonus Problem**) Problem 5.6.14. Show that every number is an eigenvalue for $Tf(x) = df/dx$, but the transformation $Tf(x) = \int_0^x f(t)dt$ has no eigenvalues (here $-\infty < x < \infty$).

Proof. For the first T, note that, if $f(x) = e^{ax}$ for any real number a, then

$$Tf(x) = \frac{df}{dx} = ae^{ax} = af(x).$$

Hence, any real number a is an eigenvalue of T.

Turning to the second T, suppose we had that $Tf(x) = af(x)$ for some number a and some function f. Then, by the definition of T,

$$\int_0^x f(t)dt = af(x).$$

Now, use the fundamental theorem of calculus to differentiate both sides:

$$f(x) = af'(x).$$

Solving for f, we see that

$$\int \frac{f'(x)dx}{f(x)} = \int \frac{1}{a}dx,$$

so

$$\ln|f(x)| = \frac{x}{a} + C.$$

Therefore, exponentiating both sides,

$$|f(x)| = e^{x/a+C} = e^C e^{x/a}.$$

We can get rid of the absolute value signs by substituting A for e^C (allowing A to possibly be negative):

$$f(x) = Ae^{x/a}.$$

Therefore, we know that

$$Tf(x) = \int_0^x f(t)dt = \int_0^x Ae^{t/a}dt = aAe^{t/a}\Big]_0^x = aAe^{x/a} - aA = a(Ae^{x/a} - A) = a(f(x) - A).$$

On the other hand, our initial assumption was that $Tf(x) = af(x)$, so it must be the case that

$$af(x) = a(f(x) - A) = af(x) - aA.$$

Hence, either $a = 0$ or $A = 0$. However, either implies that $f(x) = 0$, so T has no eigenvalues. □