

## Math 215 HW #10 Solutions

1. Problem 5.2.14. Suppose the eigenvector matrix  $S$  has  $S^T = S^{-1}$ . Show that  $A = SAS^{-1}$  is symmetric and has orthogonal eigenvectors.

*Proof.* Suppose  $S = [\vec{v}_1 \ \dots \ \vec{v}_n]$ , where  $\vec{v}_i$  are the eigenvectors of  $A$ . Then, since  $S^T = S^{-1}$ , we know that

$$I = S^T S = \begin{bmatrix} -\vec{v}_1^T & \dots & -\vec{v}_n^T \end{bmatrix} \begin{bmatrix} | & \dots & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & \dots & | \end{bmatrix} = \begin{bmatrix} \langle \vec{v}_1, \vec{v}_1 \rangle & \dots & \langle \vec{v}_1, \vec{v}_n \rangle \\ \vdots & & \vdots \\ \langle \vec{v}_n, \vec{v}_1 \rangle & \dots & \langle \vec{v}_n, \vec{v}_n \rangle \end{bmatrix},$$

so we see that  $\langle \vec{v}_i, \vec{v}_j \rangle = 0$  unless  $i = j$ ; hence, the eigenvectors of  $A$  are orthogonal.

Also, again using the fact that  $S^T = S^{-1}$ , we know that

$$A^T = (SAS^{-1})^T = (S\Lambda S^T)^T = (S^T)^T \Lambda^T S^T = S\Lambda S^T = A$$

since  $\Lambda^T = \Lambda$  and  $(S^T)^T = S$ , so we see that  $A$  is symmetric. □

2. Problem 5.2.30. Find  $\Lambda$  and  $S$  to diagonalize  $A$  in Problem 29 ( $A = \begin{bmatrix} .6 & .4 \\ .4 & .6 \end{bmatrix}$ ). What is the limit of  $\Lambda^k$  as  $k \rightarrow \infty$ ? What is the limit of  $S\Lambda^k S^{-1}$ ? In the columns of this limit matrix you see the \_\_\_\_\_.

**Answer:** To diagonalize  $A$ , we need to find the eigenvalues and eigenvectors of  $A$ . To that end, we want to solve

$$\begin{aligned} 0 = \det(A - \lambda I) &= \begin{vmatrix} .6 - \lambda & .4 \\ .4 & .6 - \lambda \end{vmatrix} \\ &= (.6 - \lambda)^2 - .4^2 \\ &= \lambda^2 - 1.2\lambda + .2 \\ &= (\lambda - 1)(\lambda - .2), \end{aligned}$$

so the eigenvalues of  $A$  are  $\lambda_1 = 1$  and  $\lambda_2 = .2$ . The eigenvector associated to  $\lambda_1 = 1$  is the generator of the nullspace of

$$A - I = \begin{bmatrix} -.4 & .4 \\ .4 & -.4 \end{bmatrix},$$

which row-reduces to  $\begin{bmatrix} -.4 & .4 \\ 0 & 0 \end{bmatrix}$ , so the nullspace consists of multiples of  $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .

On the other hand, the eigenvector associated to  $\lambda_2 = .2$  is the generator of the nullspace of

$$A - .2I = \begin{bmatrix} .4 & .4 \\ .4 & .4 \end{bmatrix},$$

which row-reduces to  $\begin{bmatrix} .4 & .4 \\ 0 & 0 \end{bmatrix}$ , so the nullspace consists of multiples of  $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ .

Therefore, we see that  $A = S\Lambda S^{-1}$  where

$$S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & .2 \end{bmatrix}.$$

Since

$$\Lambda^k = \begin{bmatrix} 1 & 0 \\ 0 & .2 \end{bmatrix}^k = \begin{bmatrix} 1^k & 0 \\ 0 & (.2)^k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{5^k} \end{bmatrix},$$

we see that  $\Lambda^k \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  as  $k \rightarrow \infty$ .

Therefore,

$$\begin{aligned} A^k &= S\Lambda^k S^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{5^k} \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2 \cdot 5^k} & \frac{1}{2 \cdot 5^k} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} + \frac{1}{2 \cdot 5^k} & \frac{1}{2} - \frac{1}{2 \cdot 5^k} \\ \frac{1}{2} - \frac{1}{2 \cdot 5^k} & \frac{1}{2} + \frac{1}{2 \cdot 5^k} \end{bmatrix}. \end{aligned}$$

Hence,

$$A^k \rightarrow \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

as  $k \rightarrow \infty$ . Each column of this matrix is an eigenvector of  $A$  corresponding to the eigenvalue  $\lambda_1 = 1$  of  $A$ .

3. Problem 5.2.32. Diagonalize  $A$  and compute  $S\Lambda^k S^{-1}$  to prove this formula for  $A^k$ :

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{has} \quad A^k = \frac{1}{2} \begin{bmatrix} 3^k + 1 & 3^k - 1 \\ 3^k - 1 & 3^k + 1 \end{bmatrix}.$$

**Answer:** To diagonalize  $A$ , first find the eigenvalues:

$$\begin{aligned} 0 &= \det(A - \lambda I) = \begin{vmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{vmatrix} \\ &= (2 - \lambda)^2 - 1 \\ &= \lambda^2 - 4\lambda + 3 \\ &= (\lambda - 3)(\lambda - 1), \end{aligned}$$

so the eigenvalues of  $\lambda_1 = 3$  and  $\lambda_2 = 1$ . Hence, the eigenvector corresponding to  $\lambda_1 = 3$  is the generator of the nullspace of

$$A - 3I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix},$$

namely  $\vec{v}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}$ .

Likewise, the eigenvector corresponding to  $\lambda_2 = 1$  is the generator of the nullspace of

$$A - I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

namely  $\vec{v}_2 = \begin{bmatrix} -1 & 1 \end{bmatrix}$ .

Therefore,  $S$  and  $S^{-1}$  are just as in Problem 2 above, so we have that

$$\begin{aligned} A^k &= S\Lambda^k S^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}^k \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^k & 3^k \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 3^k + 1 & 3^k - 1 \\ 3^k - 1 & 3^k + 1 \end{bmatrix}, \end{aligned}$$

as desired.

4. Problem 5.2.34. Suppose that  $A = S\Lambda S^{-1}$ . Take determinants to prove that

$$\det A = \lambda_1 \cdot \lambda_2 \cdots \lambda_n = \text{product of the } \lambda\text{s}.$$

This quick proof only works when  $A$  is \_\_\_\_\_.

*Proof.* Since the determinant of a product is the product of the determinants, we know that

$$\det A = \det(S\Lambda S^{-1}) = (\det S)(\det \Lambda)(\det S^{-1}).$$

In turn, since  $\det S^{-1} = \frac{1}{\det S}$ , this implies that

$$\det A = \det \Lambda.$$

However, that implies that

$$\det A = \det \Lambda = \begin{vmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{vmatrix} = \lambda_1 \cdots \lambda_n,$$

the product of the eigenvalues of  $A$ .

Clearly, this proof only works when  $A$  is diagonalizable. □

5. Problem 5.2.36. If  $A = S\Lambda S^{-1}$ , diagonalize the block matrix  $B = \begin{bmatrix} A & 0 \\ 0 & 2A \end{bmatrix}$ . Finds its eigenvalue and eigenvector matrices.

**Answer:** Notice that, since  $A = S\Lambda S^{-1}$ , we also have that  $2A = S(2\Lambda)S^{-1}$  is a diagonalization of  $2A$ . Then it's straightforward to check that

$$B = \begin{bmatrix} A & 0 \\ 0 & 2A \end{bmatrix} = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & 2\Lambda \end{bmatrix} \begin{bmatrix} S^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix}.$$

Since both  $\Lambda$  and  $2\Lambda$  are diagonal, this gives a diagonalization of  $B$ . Hence, the eigenvalue matrix is

$$\begin{bmatrix} \Lambda & 0 \\ 0 & 2\Lambda \end{bmatrix}$$

and the eigenvector matrix is

$$\begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}.$$

6. Problem 5.3.14. Multinational companies in the Americas, Asia, and Europe have assets of \$4 trillion. At the start, \$2 trillion are in the Americas and \$2 trillion in Europe. Each year  $\frac{1}{2}$  the American money stays home, and  $\frac{1}{4}$  goes to each of Asia and Europe. For Asia and Europe,  $\frac{1}{2}$  stays home and  $\frac{1}{2}$  is sent to the Americas.

(a) Find the matrix that gives

$$\begin{bmatrix} \text{Americas} \\ \text{Asia} \\ \text{Europe} \end{bmatrix}_{\text{year } k+1} = A \begin{bmatrix} \text{Americas} \\ \text{Asia} \\ \text{Europe} \end{bmatrix}_{\text{year } k}.$$

**Answer:** Given the above description of the transition, it's easy to see that

$$A = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/4 & 1/2 & 0 \\ 1/4 & 0 & 1/2 \end{bmatrix}.$$

(b) Find the eigenvalues and eigenvectors of  $A$ .

**Answer:** To find the eigenvalues, solve

$$\begin{aligned} 0 = \det(A - \lambda I) &= \begin{vmatrix} \frac{1}{2} - \lambda & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} - \lambda & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} - \lambda \end{vmatrix} \\ &= \left(\frac{1}{2} - \lambda\right) \begin{vmatrix} \frac{1}{2} - \lambda & 0 \\ 0 & \frac{1}{2} - \lambda \end{vmatrix} - \frac{1}{2} \begin{vmatrix} \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} - \lambda \end{vmatrix} + \frac{1}{2} \begin{vmatrix} \frac{1}{4} & \frac{1}{2} - \lambda \\ \frac{1}{4} & 0 \end{vmatrix} \\ &= \left(\frac{1}{2} - \lambda\right)^3 - \frac{1}{8} \left(\frac{1}{2} - \lambda\right) - \frac{1}{8} \left(\frac{1}{2} - \lambda\right) \\ &= -\lambda^3 + \frac{3}{2}\lambda^2 - \frac{1}{2}\lambda \\ &= -\lambda \left(\lambda^2 - \frac{3}{2}\lambda + \frac{1}{2}\right) \\ &= -\lambda \left(\lambda - \frac{1}{2}\right) (\lambda - 1) \end{aligned}$$

so the three eigenvalues of  $A$  are

$$\lambda_1 = 1, \quad \lambda_2 = \frac{1}{2}, \quad \lambda_3 = 0.$$

The eigenvector corresponding to  $\lambda_1 = 1$  is the generator of the nullspace of

$$A - I = \begin{bmatrix} -1/2 & 1/2 & 1/2 \\ 1/4 & -1/2 & 0 \\ 1/4 & 0 & -1/2 \end{bmatrix},$$

which row-reduces to

$$\begin{bmatrix} -1/2 & 0 & 1 \\ 0 & -1/4 & 1/4 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the nullspace consists of multiples of  $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ , so this is the eigenvector

corresponding to the eigenvalue  $\lambda_1 = 1$ .

The eigenvector corresponding to  $\lambda_2 = \frac{1}{2}$  is the generator of the nullspace of

$$A - \frac{1}{2}I = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 0 & 0 \\ 1/4 & 0 & 0 \end{bmatrix},$$

which row-reduces to

$$\begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the nullspace consists of multiples of  $\vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ , so this is the eigenvector

corresponding to the eigenvalue  $\lambda_2 = \frac{1}{2}$ .

The eigenvector corresponding to  $\lambda_3 = 0$  is the generator of the nullspace of

$$A = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/4 & 1/2 & 0 \\ 1/4 & 0 & 1/2 \end{bmatrix},$$

which row-reduces to

$$\begin{bmatrix} 1/2 & 0 & 1 \\ 0 & 1/4 & -1/4 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the nullspace consists of multiples of  $\vec{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ , so this is the eigenvector

corresponding to the eigenvalue  $\lambda_3 = 0$ .

- (c) Find the limiting distribution of the \$4 trillion as the world ends.

**Answer:** In year  $k$  the distribution is given by

$$c_1 \lambda_1^k \vec{v}_1 + c_2 \lambda_2^k \vec{v}_2 + c_3 \lambda_3^k \vec{v}_3$$

for some  $c_1, c_2, c_3$ . Since  $\lambda_3^k = 0^k = 0$  for any  $k > 0$  and since  $\lambda_2^k = \frac{1}{2^k} \rightarrow 0$  as  $k \rightarrow \infty$ , we only have to worry about the first term in the above.

Since  $\lambda_1^k = 1^k = 1$  for all  $k$ , the first term simplifies as

$$c_1 \vec{v}_1 = c_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix},$$

so, as the world ends, the distribution will be a multiple of  $\vec{v}_1$ . In fact, since the entries in  $\vec{v}_1$  add up to four, we see that  $c_1 = 1$ , so the limiting distribution has \$2 trillion in the Americas, \$1 trillion in Asia and \$1 trillion in Europe.

- (d) Find the distribution of the \$4 trillion at year  $k$ .

**Answer:** We know that there are constants  $c_1, c_2, c_3$  so that, in year  $k$ , the distribution is given by

$$\begin{aligned} c_1 \lambda_1^k \vec{v}_1 + c_2 \lambda_2^k \vec{v}_2 + c_3 \lambda_3^k \vec{v}_3 &= c_1 1^k \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + c_2 \left(\frac{1}{2}\right)^k \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + c_3 0^k \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \\ &= c_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \frac{c_2}{2^k} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}. \end{aligned}$$

In particular, when  $k = 0$ , we have that

$$\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2c_1 \\ c_1 - c_2 \\ c_1 + c_2 \end{bmatrix},$$

so  $c_1 = c_2 = 1$ . Therefore, the distribution in year  $k$  is

$$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2^k} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 - \frac{1}{2^k} \\ 1 + \frac{1}{2^k} \end{bmatrix}.$$

In other words, in year  $k$ , the Americas have \$2 trillion, Asia has  $\$(1 - \frac{1}{2^k})$  trillion and Europe has  $\$(1 + \frac{1}{2^k})$  trillion.

7. Problem 5.4.8. Suppose the rabbit population  $r$  and the wolf population  $w$  are governed by

$$\begin{aligned} \frac{dr}{dt} &= 4r - 2w \\ \frac{dw}{dt} &= r + w. \end{aligned}$$

- (a) Is this system stable, neutrally stable, or unstable?

**Answer:** Let  $\vec{p}(t) = \begin{bmatrix} r(t) \\ w(t) \end{bmatrix}$  be the population vector. Then the above system becomes

$$\vec{p}'(t) = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \vec{p}(t).$$

We want to find the eigenvalues of the matrix  $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$ . To do so, we want to solve

$$\begin{aligned} 0 = \det(A - \lambda I) &= \begin{vmatrix} 4 - \lambda & -2 \\ 1 & 1 - \lambda \end{vmatrix} \\ &= (4 - \lambda)(1 - \lambda) + 2 \\ &= \lambda^2 - 5\lambda + 6 \\ &= (\lambda - 2)(\lambda - 3). \end{aligned}$$

Therefore, the eigenvalues of  $A$  are  $\lambda_1 = 3$  and  $\lambda_2 = 2$ . Therefore, the system must be unstable, since both eigenvalues are positive real numbers.

- (b) If initially  $r = 300$  and  $w = 200$ , what are the populations at time  $t$ ?

**Answer:** Since we've already found the eigenvalues of  $A$ , the next step is to diagonalize  $A$ . To do so, we need to find the eigenvectors of  $A$ . The eigenvector corresponding to  $\lambda_1 = 3$  will be the generator of the nullspace of

$$A - 3I = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}.$$

The nullspace consists of multiples of the vector  $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ , so this is the eigenvector corresponding to  $\lambda_1 = 3$ .

The eigenvector corresponding to  $\lambda_2 = 2$  will be the generator of the nullspace of

$$A - 2I = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}.$$

The nullspace consists of multiples of the vector  $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ , so this is the eigenvector corresponding to  $\lambda_2 = 2$ .

Thus,

$$S = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

is the eigenvector matrix of  $A$ , and

$$\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

is the eigenvalue matrix.

Since  $\det S = 2 - 1 = 1$ , we see that

$$S^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix},$$

so

$$A = S\Lambda S^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

Therefore,

$$e^{At} = Se^{\Lambda t}S^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix},$$

so we have that

$$\begin{aligned} \vec{p}(t) = e^{At}\vec{p}(0) &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 300 \\ 200 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 100 \\ 100 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 100e^{3t} \\ 100e^{2t} \end{bmatrix} \\ &= \begin{bmatrix} 200e^{3t} + 100e^{2t} \\ 100e^{3t} + 100e^{2t} \end{bmatrix}. \end{aligned}$$

Therefore, the population of rabbits at time  $t$  is  $r(t) = 200e^{3t} + 100e^{2t}$ , whereas the population of wolves at time  $t$  is  $w(t) = 100e^{3t} + 100e^{2t}$ .

- (c) After a long time, what is the proportion of rabbits to wolves?

**Answer:** When  $t$  is very large,  $e^{3t}$  is much larger than  $e^{2t}$ , so only the  $e^{3t}$  terms will matter in the above expression for  $\vec{p}(t)$ . Therefore, for large  $t$  the population vector is approximately

$$\begin{bmatrix} 200e^{3t} \\ 100e^{3t} \end{bmatrix} = e^{3t} \begin{bmatrix} 200 \\ 100 \end{bmatrix},$$

so there are twice as many rabbits as wolves.

8. Problem 5.4.36. Write  $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$  in the form  $S\Lambda S^{-1}$ . Find  $e^{At}$  from  $Se^{\Lambda t}S^{-1}$ .

**Answer:** We want to diagonalize  $A$ , so the first step is to find the eigenvalues of  $A$ . Thus, we want to solve

$$\begin{aligned} 0 = \det(A - \lambda I) &= \begin{vmatrix} 1 - \lambda & 1 \\ 0 & -\lambda \end{vmatrix} \\ &= (1 - \lambda)(-\lambda) - 0(1) \\ &= \lambda(\lambda - 1), \end{aligned}$$

so the eigenvalues of  $A$  are  $\lambda_1 = 1$  and  $\lambda_2 = 0$ . Then the eigenvector associated to  $\lambda_1 = 1$  is the generator of the nullspace of

$$A - I = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}.$$



Hence, the nullspace consists of multiples of the vector  $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ , so this is the eigenvector associated to  $\lambda_1 = 1$ .

On the other hand, the eigenvector associated to  $\lambda_2 = 0$  is the generator of the nullspace of

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

This nullspace consists of multiples of the vector  $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ , so this is the eigenvector associated to  $\lambda_2 = 0$ .

Therefore, the eigenvector matrix is

$$S = [\vec{v}_1 \ \vec{v}_2] = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

and the eigenvalue matrix is

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since  $\det S = 1$ , we can see that

$$S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

so we have that

$$A = S\Lambda S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} e^{At} &= S e^{\Lambda t} S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{0t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & e^t \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^t & e^t - 1 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

9. (**Bonus Problem**) Problem 5.4.42. Find a solution  $x(t)$ ,  $y(t)$  of the first system that gets large as  $t \rightarrow \infty$ . To avoid this instability a scientist thought of exchanging the two equations!

$$\begin{array}{ll} \begin{array}{l} dx/dt = 0x - 4y \\ dy/dt = -2x + 2y \end{array} & \text{becomes} \quad \begin{array}{l} dy/dt = -2x + 2y \\ dx/dt = 0x - 4y. \end{array} \end{array}$$

Now the matrix  $\begin{bmatrix} -2 & 2 \\ 0 & -4 \end{bmatrix}$  is stable. It has  $\lambda < 0$ . Comment on this craziness.

**Answer:** Let

$$\vec{z}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

Then we can write the above system of equations as

$$\vec{z}'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0x(t) - 4y(t) \\ -2x(t) + 2y(t) \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A\vec{z}(t).$$

In other words, if  $A = \begin{bmatrix} 0 & -4 \\ -2 & 2 \end{bmatrix}$ , then  $\vec{z}'(t) = A\vec{z}(t)$ .

Therefore, to solve for  $\vec{z}(t)$ , we first want to diagonalize  $A$ . The eigenvalues of  $A$  are the solutions to the equation

$$\begin{aligned} 0 = \det(A - \lambda I) &= \begin{vmatrix} -\lambda & -4 \\ -2 & 2 - \lambda \end{vmatrix} \\ &= -\lambda(2 - \lambda) - 8 \\ &= \lambda^2 - 2\lambda - 8 \\ &= (\lambda - 4)(\lambda + 2), \end{aligned}$$

so the eigenvalues of  $\lambda_1 = 4$  and  $\lambda_2 = -2$ . The eigenvector corresponding to  $\lambda_1 = 4$  will be the generator of the nullspace of

$$A - 4I = \begin{bmatrix} -4 & -4 \\ -2 & -2 \end{bmatrix}.$$

The nullspace of the matrix consists of multiples of  $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ , so this is the eigenvector corresponding to  $\lambda_1 = 4$ .

On the other hand, the eigenvector corresponding to  $\lambda_2 = -2$  will be the generator of the nullspace of

$$A + 2I = \begin{bmatrix} 2 & -4 \\ -2 & 4 \end{bmatrix}.$$

This nullspace is generated by multiples of  $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ , so this is the eigenvector corresponding to the eigenvalue  $\lambda_2 = -2$ .

Thus, we see that the eigenvector matrix for  $A$  is

$$S = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}$$

and the eigenvalue matrix is

$$\Lambda = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}.$$

Since  $\det S = -3$ , we see that

$$S^{-1} = \frac{1}{-3} \begin{bmatrix} 1 & -2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1/3 & 2/3 \\ 1/3 & 1/3 \end{bmatrix}$$

and

$$A = S\Lambda S^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -1/3 & 2/3 \\ 1/3 & 1/3 \end{bmatrix}.$$

In turn, solutions to the system of differential equations take the form

$$\begin{aligned} \vec{z}(t) &= e^{At} \vec{z}(0) = S e^{\Lambda t} S^{-1} \vec{z}(0) \\ &= \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} -1/3 & 2/3 \\ 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} \\ &= \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} -\frac{1}{3}x(0) + \frac{2}{3}y(0) \\ \frac{1}{3}x(0) + \frac{1}{3}y(0) \end{bmatrix} \\ &= \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \left(-\frac{1}{3}x(0) + \frac{2}{3}y(0)\right) e^{4t} \\ \left(\frac{1}{3}x(0) + \frac{1}{3}y(0)\right) e^{-2t} \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{1}{3}x(0) - \frac{2}{3}y(0)\right) e^{4t} + \left(\frac{2}{3}x(0) + \frac{2}{3}y(0)\right) e^{-2t} \\ \left(-\frac{1}{3}x(0) + \frac{2}{3}y(0)\right) e^{4t} + \left(\frac{1}{3}x(0) + \frac{1}{3}y(0)\right) e^{-2t} \end{bmatrix} \end{aligned}$$

As long as  $x(0)$  and  $y(0)$  are not both equal to zero, we get a solution which gets large as  $t \rightarrow \infty$ .

Now, we flip the equations:

$$\begin{aligned} dy/dt &= -2x + 2y \\ dx/dt &= 0x - 4y. \end{aligned}$$

The matrix  $\begin{bmatrix} -2 & 2 \\ 0 & -4 \end{bmatrix}$  is stable (it has eigenvalues  $-2$  and  $-4$ ). However, this matrix has nothing to do with the above system, so there's no conflict with the above reasoning. To see that this is not the right matrix for this system, we write the system out as a matrix equation:

$$\begin{bmatrix} y'(t) \\ x'(t) \end{bmatrix} = \begin{bmatrix} -2x(t) + 2y(t) \\ 0x(t) - 4y(t) \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ x(t) \end{bmatrix}.$$

Therefore, the relevant matrix is

$$A = \begin{bmatrix} 2 & -2 \\ -4 & 0 \end{bmatrix},$$

which you can check is unstable (it has eigenvalues  $4$  and  $-2$ , just as in the original system).