Math 115 HW #9 Solutions

1. Solve the differential equation

$$\sin x \frac{dy}{dx} + (\cos x) y = x \sin(x^2).$$

Answer: Dividing everything by $\sin x$ yields the linear first order equation

$$y' + \frac{\cos x}{\sin x} y = \frac{x \sin(x^2)}{\sin x}.$$

Here $P(x) = \frac{\cos x}{\sin x}$ and $Q(x) = \frac{x \sin(x^2)}{\sin x}$, so

$$\int P(x) dx = \int \frac{\cos x}{\sin x} dx = \ln|\sin x|$$

(doing a u-substitution with $u = \sin x$). Thus,

$$\mu(x) = e^{\int P(x) dx} = e^{\ln|sin x|} = |\sin x|.$$

We can drop the absolute value signs (they would cancel in the final expression for y, anyway) and let $\mu(x) = \sin x$. Then

$$\int \mu(x) Q(x) dx = \int \sin x \frac{x \sin(x^2)}{\sin x} dx = \int x \sin(x^2) dx = \frac{1}{2}(-\cos(x^2) + C) = -\frac{1}{2}\cos(x^2) + C'$$

(by a u-substitution with $u = x^2$). Therefore,

$$y = \frac{1}{\mu(x)} \int \mu(x) Q(x) dx = \frac{1}{\sin x}\left(-\frac{1}{2}\cos(x^2) + C'\right) = -\frac{1}{2}\frac{\cos(x^2)}{\sin x} + \frac{C'}{\sin x}.$$

2. Solve the initial-value problem

$$y' = x + y, \quad y(0) = 2.$$

Answer: Re-write in the standard form for a linear equation:

$$y' - y = x.$$

Here $P(x) = -1$ and $Q(x) = x$, so

$$\int P(x) dx = \int -dx = -x.$$

Hence

$$\mu(x) = e^{\int P(x) dx} = e^{-x}.$$

In turn,

$$\int \mu(x) Q(x) dx = \int e^{-x} x dx = -xe^{-x} - e^{-x} + C$$

(integrating by parts with $u = x$, $dv = e^{-x}dx$). Therefore,

$$y = \frac{1}{\mu(x)} \int \mu(x)Q(x)dx = \frac{1}{e^{-x}} \left(-xe^{-x} - e^{-x} + C\right) = e^{x} \left(-xe^{-x} - e^{-x} + C\right) = -x-1+Ce^{x}.$$

Therefore, plugging in $x = 0$, we see that

$$2 = -(0) - 1 + Ce^{0} = -1 + C,$$

so $C = 3$. Thus, the solution to the initial-value problem is

$$y = -x - 1 + 3e^{x}.$$

3. Solve the initial-value problem

$$xy' = y + x^2 \sin x, \quad y(\pi) = 0.$$

Answer: Re-write as

$$xy' - y = x^2 \sin x.$$

Then, after dividing everything by x, we get the standard form of a linear first-order equation:

$$y' - \frac{1}{x}y = x \sin x.$$

Here $P(x) = -\frac{1}{x}$ and $Q(x) = x \sin x$, so

$$\int P(x)dx = \int -\frac{1}{x}dx = -\ln|x| = \ln\left(\frac{1}{|x|}\right).$$

Thus,

$$\mu(x) = e^{\int P(x)dx} = e^{\ln\left(\frac{1}{|x|}\right)} = \frac{1}{|x|}.$$

In the statement of the problem I should have specified that $x > 0$, so we can ignore the absolute value signs and let $\mu(x) = \frac{1}{x}$. Therefore,

$$\int \mu(x)Q(x)dx = \int \frac{1}{x}x \sin x dx = \int \sin x dx = -\cos x + C.$$

Therefore,

$$y = \frac{1}{\mu(x)} \int \mu(x)Q(x)dx = \frac{1}{\frac{1}{x}} (-\cos x + C) = -x \cos x + Cx.$$

Plugging in $x = \pi$ yields

$$0 = -\pi \cos \pi + C\pi,$$

so

$$C\pi = \pi \cos \pi = -\pi,$$

meaning that $C = -1$. Thus, the solutions to the initial-value problem is

$$y = -x \cos x - x.$$

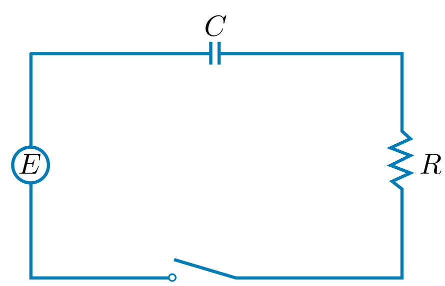
4. The figure shows a circuit containing an electromotive force, a capacitor with a capacitance of C farads (F), and a resistor with a resistance of R ohms (Ω). Ohm's Law says that the voltage drop across the resistor is RI. The voltage drop across the capacitor is Q/C, where Q is the charge (in coulombs), so in this case Kirchhoff's Law gives

$$RI + \frac{Q}{C} = E(t).$$

By definition $I = dQ/dt$, so we have

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E(t).$$

Suppose the resistance is $5\,\Omega$, the capacitance is $0.05\,F$, a battery gives a constant voltage of $60\,V$, and the initial charge is $Q(0) = 0$ coulombs. Find the charge and the current at time t.



Answer: Plugging in the values for R, C, and E, we see that

$$5\frac{dQ}{dt} + \frac{1}{0.05}Q = 60,$$

or

$$5\frac{dQ}{dt} + 20Q = 60.$$

Dividing everything by 5 yields the linear equation in standard form

$$\frac{dQ}{dt} + 4Q = 12.$$

Here $P(t) = 4$ and $Q(t) = 12$, so

$$\int P(t)dt = \int 4dt = 4t.$$

Thus,

$$\mu(t) = e^{\int P(t)dt} = e^{4t}.$$

In turn,

$$\int \mu(t)Q(t)dt = \int e^{4t}12dt = 3\int e^{4t}4dt = 3\left(e^{4t} + C\right).$$

Therefore,

$$Q(t) = \frac{1}{\mu(t)}\int \mu(t)Q(t)dt = 3 + \frac{C'}{e^{4t}}.$$

Using what we know about the situation when $t = 0$, we can solve for C':

$$0 = 3 + \frac{C'}{e^{4 \cdot 0}} = 3 + C',$$

so $C' = -3$.

Hence,

$$Q(t) = 3 - \frac{3}{e^{4t}}.$$

5. An object with mass m is dropped from rest and we assume that the air resistance is proportional to the speed of the object. If $s(t)$ is the distance dropped after t seconds, then the speed is $v = s'(t)$ and the acceleration is $a = v'(t)$. If g is the acceleration due to gravity, then the downward force on the object is $mg - cv$, where c is a positive constant, and Newton's Second Law gives

$$m\frac{dv}{dt} = mg - cv.$$

(a) Solve this for v (note that this is a linear differential equation in v).

Answer: Re-write as

$$m\frac{dv}{dt} + cv = mg.$$

Then dividing everything by m yields the linear equation in standard form

$$\frac{dv}{dt} + \frac{c}{m}v = g.$$

Here $P(t) = \frac{c}{m}$ and $Q(t) = g$, so

$$\int P(t)dt = \int \frac{c}{m}dt = \frac{c}{m}t.$$

Thus,

$$\mu(t) = e^{\int P(t)dt} = e^{\frac{c}{m}t}.$$

In turn, this means that

$$\int \mu(t)Q(t)dt = \int e^{\frac{c}{m}t}gdt = \frac{mg}{c}e^{\frac{c}{m}t} + C.$$

Therefore,

$$v(t) = \frac{1}{\mu(t)}\int \mu(t)Q(t)dt = \frac{1}{e^{\frac{c}{m}t}}\left(\frac{mg}{c}e^{\frac{c}{m}t} + C\right) = \frac{mg}{c} + Ce^{-\frac{c}{m}t}.$$

Since the object starts at rest, $v(0) = 0$, so

$$0 = \frac{mg}{c} + Ce^{\frac{c}{m}\cdot 0} = \frac{mg}{c} + C.$$

Hence, $C = -\frac{mg}{c}$, so

$$v(t) = \frac{mg}{c} - \frac{mg}{c}e^{-\frac{c}{m}t} = \frac{mg}{c}\left(1 - e^{-\frac{c}{m}t}\right).$$

(b) What is the terminal velocity of the falling object?

Answer: We can solve for the terminal velocity by computing

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \frac{mg}{c}\left(1 - e^{-\frac{c}{m}t}\right) = \frac{mg}{c}\lim_{t\to\infty}\left(1 - e^{-\frac{c}{m}t}\right) = \frac{mg}{c}(1-0) = \frac{mg}{c}.$$

Hence, the terminal velocity of the falling object is $\frac{mg}{c}$.

(c) Find the distance the object has fallen after t seconds.

Answer: Remember that $v(t) = s'(t)$, so $s(t) = \int_0^t v(\tau)d\tau$. Thus, we compute that the distance traveled after t seconds is

$$s(t) = \int_0^t v(\tau)d\tau = \int_0^t \frac{mg}{c}\left(1 - e^{-\frac{c}{m}\tau}\right)d\tau = \frac{mg}{c}\left[\tau + \frac{m}{c}e^{-\frac{c}{m}\tau}\right]_0^t = \frac{mg}{c}t + \frac{m^2 g}{c^2}e^{-\frac{c}{m}t} - \frac{m^2 g}{c^2}.$$

6. Each system of differential equations is a model for two species that either compete for the same resources or cooperate for mutual benefit (flowering plants and insect pollinators, for instance). Decide whether each system describes competition or cooperation and explain why it is a reasonable model. (*Hint*: think about the effect an increase in one species has on the growth rate of the other.)

(a) $\frac{dx}{dt} = 0.12x - 0.0006x^2 + 0.00001xy$

$\frac{dy}{dt} = 0.08y + 0.00004xy$

Answer: Increasing the y population causes an increase in $\frac{dx}{dt}$, the rate of growth of x. Likewise, an increase in the x population causes an increase in $\frac{dy}{dt}$, the rate of growth of y. Therefore, this system of equations describes the populations of two species which are cooperating with each other.

(b) $\frac{dx}{dt} = 0.15x - 0.0002x^2 - 0.0006xy$

$\frac{dy}{dt} = 0.2y - 0.00008y^2 - 0.0002xy$

Answer: Increasing the y population causes a decrease in the rate of growth of x, and likewise an increase in the x population causes a decrease in the rate of growth of the y population. Thus, these equations describe the populations of two species which are in competition with each other.

7. Populations of aphids and ladybugs are modeled by the equations

$$\frac{dA}{dt} = 2A - 0.01AL$$
$$\frac{dL}{dt} = -0.5L + 0.0001AL$$

(a) Find the equilibrium solutions and explain their significance.

Answer: An equilibrium solution is a solutions for which both A and L are constant. If A is constant, then $\frac{dA}{dt} = 0$, so the first equation becomes

$$0 = 2A - 0.01AL = A(2 - 0.01L),$$

meaning either $A = 0$ or $L = 200$. If $A = 0$ and L is constant, then $\frac{dL}{dt} = 0$, so the second equation reduces to

$$0 = -0.5L + 0.0001(0)L = -0.5L,$$

so $L = 0$. Thus, $(A, L) = (0, 0)$ is one equilibrium solution. On the other hand, if $L = 200$ and A is constant, then

$$0 = -0.5(200) + 0.0001A(200) = -100 + 0.02A,$$

so $A = 5000$. Therefore, $(A, L) = (5000, 200)$ is the other equilibrium solution.

The trivial solution $A = L = 0$ just says that, if there are no aphids and no ladybugs to start out with, there will continue to be none forever.

The solution $(A, L) = (5000, 200)$ says that when there are 5000 aphids and 200 ladybugs, the populations are just the right size that there are no changes in the size of either population.

(b) Find an expression for dL/dA.

Answer: If L were a function of A, then we would have that

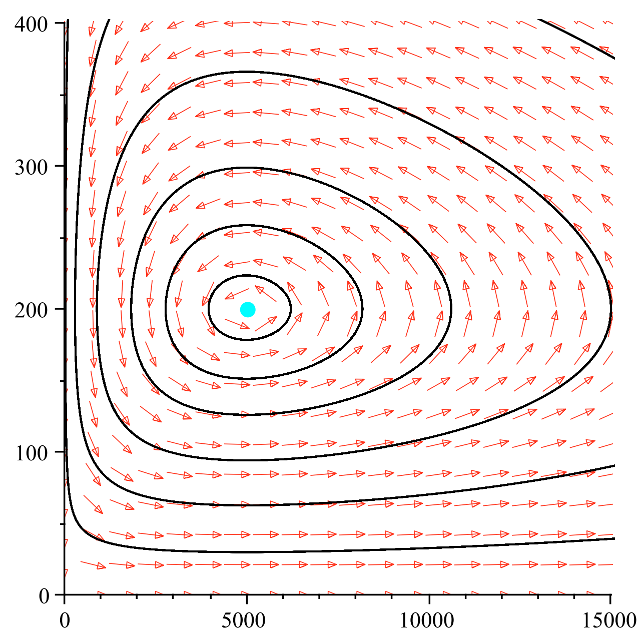
$$\frac{dL}{dt} = \frac{dL}{dA}\frac{dA}{dt}.$$

Therefore,

$$\frac{dL}{dA} = \frac{\frac{dL}{dt}}{\frac{dA}{dt}} = \frac{-0.5L + 0.0001AL}{2A - 0.01AL}.$$

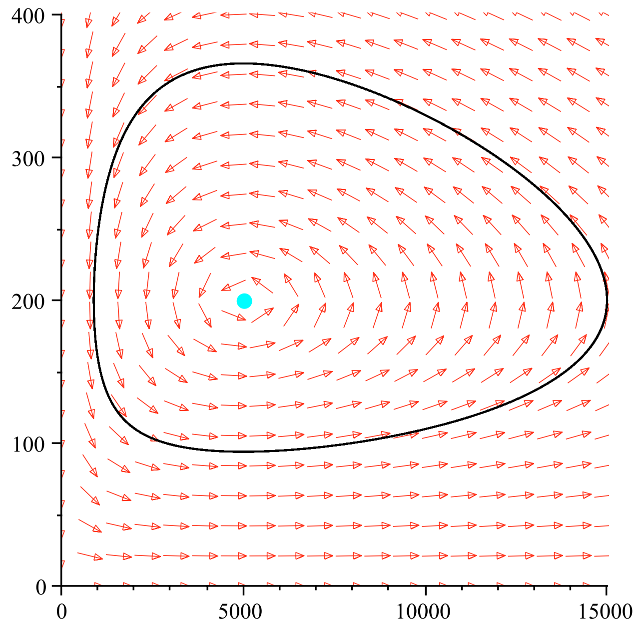
(c) The direction field for the differential equation in part (b) is shown [omitted from this answer key]. Use it to sketch a phase portrait.

Answer: Below is a phase portrait of the system:



(d) Suppose that at time $t = 0$ there are 1000 aphids and 200 ladybugs. Draw the corresponding phase trajectory and use it to describe how both populations change.

Answer: Below is the phase trajectory determined by this initial condition:



This says that in the beginning there aren't enough aphids to support 200 ladybugs, so the ladybug population decreases rapidly, while the aphid population increases very slowly. As more ladybugs die, the rate of decrease of the population of ladybugs slows, while the rate of growth of the aphid population increases.

After the ladybug population bottoms out at 100, the ladybug population starts increasing; the aphid population is still growing rapidly, eventually getting to a maximum of almost 15,000 aphids. Of course, with all these aphids to eat, the ladybug population increases rapidly, eventually causing the aphids to start dying off.

The ladybugs reach a maximum population of more than 350, but by this point the falling aphid population means that this is unsustainable. As the aphid population bottoms out at 1000, the ladybug population falls through 200, starting the cycle anew.

(e) Use part (d) to make rough sketches of the aphid and ladybug populations as functions of t. How are the graphs related to each other?

Answer: Both graphs have the same period and the graph of L peaks about a quarter of a cycle after the graph of A.