Math 115 HW #7 Solutions

1. Write the number
$$\frac{3+2i}{4-3i}$$
in the form $a+bi$.

Answer: Multiply numerator and denominator by the conjugate of the denominator:
$$\frac{3+2i}{4-3i}\cdot\frac{4+3i}{4+3i} = \frac{(3+2i)(4+3i)}{4^2+3^2} = \frac{12+8i+9i+6i^2}{25} = \frac{(12-6)+i(8+9)}{25} = \frac{6}{25}+\frac{17}{25}i.$$

2. Prove the following properties of complex numbers

(a) $\overline{z+w} = \overline{z}+\overline{w}$

Proof. Let $z = a+bi$ and let $w = c+di$. Then
$$\overline{z}+\overline{w} = (a-bi)+(c-di) = (a+c)-i(b+d).$$
On the other hand,
$$z+w = (a+bi)+(c+di) = (a+c)+i(b+d),$$
so
$$\overline{z+w} = (a+c)-i(b+d) = \overline{z}+\overline{w},$$
as we saw above. □

(b) $\overline{zw} = \bar{z}\bar{w}$

Proof. Let $z = a+bi$ and $w = c+di$. Then
$$\bar{z}\bar{w} = (a-bi)(c-di) = ac-bci-adi+bdi^2 = (ac-bd)-i(bc+ad).$$
On the other hand,
$$zw = (a+bi)(c+di) = ac+bci+adi+bdi^2 = (ac-bd)+i(bc+ad),$$
so
$$\overline{zw} = (ac-bd)-i(bc+ad) = \bar{z}\bar{w},$$
as we saw above. □

3. Find all solutions of the equation
$$2x^2-2x+1=0.$$

Answer: By the quadratic formula, solutions to this equation are given by
$$x = \frac{2\pm\sqrt{(-2)^2-4(2)(1)}}{2(2)} = \frac{2\pm\sqrt{4-8}}{4} = \frac{1}{2}\pm\frac{\sqrt{-4}}{4}.$$
Since we can write $\sqrt{-4}$ as $\sqrt{4}i = 2i$, this means that the two solutions of the given equation are
$$\frac{1}{2}\pm\frac{1}{2}i.$$

4. Let
$$z = \sqrt{3} + i, \quad w = 1 + \sqrt{3}i.$$
Find polar forms for zw, z/w and $1/z$ by first putting z and w into polar form.

Answer: Remember that, if $\zeta = x + iy$ is to be written in the polar form $\zeta = re^{i\theta}$, we know that
$$r = |\zeta|, \qquad \theta = \tan^{-1}\frac{y}{x}.$$
Therefore, for the given z and w, we can determine the polar forms by computing
$$\begin{aligned} |z| &= \sqrt{\sqrt{3}^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2 \\ |w| &= \sqrt{1^2 + \sqrt{3}^2} = \sqrt{1+3} = \sqrt{4} = 2 \\ \tan^{-1}\frac{1}{\sqrt{3}} &= \frac{\pi}{6} \\ \tan^{-1}\frac{\sqrt{3}}{1} &= \frac{\pi}{3}. \end{aligned}$$
Thus, we see that
$$z = 2e^{i(\pi/6)} \quad \text{and} \quad w = 2e^{i(\pi/3)}.$$
Therefore, using these expressions for z and w,
$$zw = \left(2e^{i(\pi/6)}\right)\left(2e^{i(\pi/3)}\right) = (2\cdot 2)e^{i(\pi/6+\pi/3)} = 4e^{i\pi/2}$$
(which is just another name for $4i$).

Similarly,
$$\frac{z}{w} = \frac{2e^{i(\pi/6)}}{2e^{i(\pi/3)}} = \frac{2}{2}e^{i(\pi/6-\pi/3)} = e^{i(-\pi/6)}.$$
Finally,
$$\frac{1}{z} = \frac{1}{2e^{i(\pi/6)}} = \frac{1}{2}e^{-i(\pi/6)} = \frac{1}{2}e^{i(-\pi/6)}.$$

5. Find all the fifth roots of 32 and sketch them in the complex plane.

Answer: Suppose α is a fifth root of 32. Write α in polar form: $\alpha = re^{i\theta}$. Then
$$32 = \alpha^5 = \left(re^{i\theta}\right)^5 = r^5e^{i(5\theta)}.$$
Therefore, since we can write 32 in polar form as
$$32e^{i(0)}, 32e^{i(2\pi)}, 32e^{i(4\pi)}, 32e^{i(6\pi)}, 32e^{i(8\pi)},$$
we see that $r^5 = 32$, meaning that $r = 2$. Also,
$$5\theta = 0, 2\pi, 4\pi, 6\pi, 8\pi,$$
so the fifth roots of 32 are
$$2e^{i(0)} = 2, 2e^{i(2\pi/5)}, 2e^{i(4\pi/5)}, 2e^{i(6\pi/5)}, 2e^{i(8\pi/5)}.$$

6. Write

$$e^{-i\pi}$$

in the form $a + bi$.

Answer: Using Euler's formula,

$$e^{-i\pi} = e^{i(-\pi)} = \cos(-\pi) + i\sin(-\pi) = -1 + 0i = -1.$$

7. Use Euler's formula (i.e. $e^{i\theta} = \cos\theta + i\sin\theta$) to prove the following formulas for $\cos x$ and $\sin x$:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

Proof. Using Euler's formula,

$$e^{ix} = \cos x + i\sin x.$$

Similarly, $e^{-ix} = \cos(-x) + i\sin(-x)$, which means (using the fact that cosine is even and sine is odd)

$$e^{-ix} = \cos x - i\sin x.$$

Therefore,

$$\frac{e^{ix} + e^{-ix}}{2} = \frac{(\cos x + i\sin x) + (\cos x - i\sin x)}{2} = \frac{2\cos x}{2} = \cos x,$$

as desired.

Likewise

$$\frac{e^{ix} - e^{-ix}}{2i} = \frac{(\cos x + i\sin x) - (\cos x - i\sin x)}{2i} = \frac{2i\sin x}{2i} = \sin x,$$

completing the proof. □