

## Math 115 HW #5 Solutions

### From §12.7

8. Does the series

$$\sum_{k=1}^{\infty} \frac{2^k k!}{(k+2)!}$$

converge or diverge?

**Answer:** Using the Ratio Test,

$$\lim_{k \rightarrow \infty} \left| \frac{\frac{2^{k+1}(k+1)!}{(k+3)!}}{\frac{2^k k!}{(k+2)!}} \right| = \lim_{k \rightarrow \infty} \frac{2(k+1)}{k+3} = 2 \lim_{k \rightarrow \infty} \frac{k+1}{k+3} = 2.$$

Since  $2 > 1$ , the Ratio Test says that the series diverges.

12. Does the series

$$\sum_{n=1}^{\infty} \sin n$$

converge or diverge?

**Answer:** The limit

$$\lim_{n \rightarrow \infty} \sin n$$

does not exist, so the Divergence Test says that the series diverges.

22. Does the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^2 - 1}}{n^3 + 2n^2 + 5}$$

converge or diverge?

**Answer:** Notice that, for any  $n$ ,

$$\sqrt{n^2 - 1} < \sqrt{n^2} = n.$$

Also,

$$n^3 + 2n^2 + 5 > n^3$$

for all  $n$ . Therefore, we see that

$$\frac{\sqrt{n^2 - 1}}{n^3 + 2n^2 + 5} < \frac{n}{n^3} = \frac{1}{n^2}$$

for all  $n$ . Hence, since the series  $\sum \frac{1}{n^2}$  converges (it's a  $p$ -series with  $p = 2 > 1$ ), the comparison test says that the given series converges.

28. Does the series

$$\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$$

converge or diverge?

**Answer:** Do a limit comparison to the convergent series  $\sum \frac{1}{n^2}$  (which converges because it's a  $p$ -series with  $p = 2 > 1$ ):

$$\lim_{n \rightarrow \infty} \frac{\frac{e^{1/n}}{n^2}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} e^{1/n} = \lim_{n \rightarrow \infty} \sqrt[n]{e} = 1.$$

Therefore, the Limit Comparison Test says that the series  $\sum \frac{e^{1/n}}{n^2}$  converges.

## From §12.8

10. Find the radius of convergence and interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{10^n x^n}{n^3}.$$

**Answer:** Using the Ratio Test,

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{10^{n+1} x^{n+1}}{(n+1)^3}}{\frac{10^n x^n}{n^3}} \right| = \lim_{n \rightarrow \infty} \frac{10|x|n^3}{(n+1)^3} = 10|x| \lim_{n \rightarrow \infty} \frac{n^3}{(n+1)^3} = 10|x|,$$

which is less than 1 when  $|x| < \frac{1}{10}$ . Thus, the radius of convergence of the power series is  $\frac{1}{10}$ .

To determine the interval of convergence, we check what happens when  $|x| = \frac{1}{10}$ . If  $x = \frac{1}{10}$ , then the series is given by

$$\sum_{n=1}^{\infty} \frac{10^n \left(\frac{1}{10}\right)^n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^3},$$

which converges (it's a  $p$ -series with  $p = 3 > 1$ ). If  $x = -\frac{1}{10}$ , then the series is given by

$$\sum_{n=1}^{\infty} \frac{10^n \left(-\frac{1}{10}\right)^n}{n^3} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3},$$

which converges (you can see this either by the Alternating Series Test or by noting that, as we already saw above, this series is absolutely convergent).

Putting this all together, we see that the interval of convergence of the power series is

$$\left[-\frac{1}{10}, \frac{1}{10}\right]$$

16. Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1}.$$

**Answer:** Using the Ratio Test,

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} \frac{(x-3)^{n+1}}{2(n+1)+1}}{(-1)^n \frac{(x-3)^n}{2n+1}} \right| = \lim_{n \rightarrow \infty} \frac{|x-3|(2n+1)}{2n+3} = |x-3| \lim_{n \rightarrow \infty} \frac{2n+1}{2n+3} = |x-3|,$$

which is less than 1 when  $|x-3| < 1$ . Hence, the radius of convergence of the series is 1.

To find the interval of convergence, note that  $|x-3| < 1$  is equivalent to

$$2 < x < 4,$$

so the endpoints of the interval are  $x = 2$  and  $x = 4$ . When  $x = 2$ ,  $x - 3 = -1$ , so the series is given by

$$\sum_{n=0}^{\infty} (-1)^n \frac{(-1)^n}{2n+1} = \sum_{n=0}^{\infty} \frac{1}{2n+3},$$

which is a divergent series (do a limit comparison to, e.g.,  $\sum \frac{1}{n}$ ). When  $x = 4$ ,  $x - 3 = 1$ , so the series is given by

$$\sum_{n=0}^{\infty} (-1)^n \frac{1^n}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1},$$

which converges by the Alternating Series Test.

Putting this all together, the interval of convergence of the power series is

$$(2, 4].$$

24. Find the radius of convergence and interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{n^2 x^n}{2 \cdot 4 \cdot 6 \cdots (2n)}.$$

**Answer:** Using the Ratio Test,

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{(n+1)^2 x^{n+1}}{2 \cdot 4 \cdots (2n)(2n+2)}}{\frac{n^2 x^n}{2 \cdot 4 \cdots (2n)}} \right| = \lim_{n \rightarrow \infty} \frac{|x|(n+1)^2}{n^2(2n+2)} = |x| \lim_{n \rightarrow \infty} \frac{(n+1)^2}{n^2(2n+2)} = |x| \lim_{n \rightarrow \infty} \frac{n^2 + 2n + 1}{2n^3 + 2n^2} = 0,$$

which is less than 1 no matter what  $x$  is. Therefore, the radius of convergence is  $\infty$  and the interval of convergence is

$$(-\infty, \infty).$$