Math 115 HW #1 Solutions

1. The ratio $\log_3(x)/\log_2(x)$ is constant, i.e., it is equal to some number C for all x. What is C?

   Answer: The quantity $\log_3(x)$ can be re-written as

$$\log_3(x) = \frac{\ln x}{\ln 3}.$$

   Likewise, $\log_2(x) = \frac{\ln x}{\ln 2}$, so

$$\frac{\log_3(x)}{\log_2(x)} = \frac{\frac{\ln x}{\ln 3}}{\frac{\ln x}{\ln 2}} = \frac{\ln 2}{\ln 3},$$

   which is a constant number approximately equal to 0.631.

2. Let $f(x) = \ln(x+1) - \ln(x)$.

   - By differentiating $f(x)$, show that $f(x)$ is a decreasing function of x.

     Answer: First, notice that the function $f(x)$ is only well-defined when $x > 0$, so we should restrict our attention to those values of x. The derivative of $f(x)$ is

$$f'(x) = \frac{1}{x+1} - \frac{1}{x} = \frac{x}{x(x+1)} - \frac{x+1}{x(x+1)} = \frac{-1}{x(x+1)}.$$

     For $x > 0$ this quantity is negative. Remember that a function is decreasing when its derivative is negative, so this implies that $f(x)$ is a decreasing function.

   - What is the limit of $f(x)$ as $x \to \infty$? (Hint: express $f(x)$ as $\ln(g(x))$, and figure out what happens to $g(x)$ as $x \to \infty$)

     Answer: We can re-write $f(x)$ as

$$f(x) = \ln(x+1) - \ln(x) = \ln\left(\frac{x+1}{x}\right). \tag{1}$$

     We're interested in determining

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \ln\left(\frac{x+1}{x}\right).$$

     Since the natural logarithm is a continuous function, we can move the limit inside:

$$\lim_{x\to\infty} \ln\left(\frac{x+1}{x}\right) = \ln\left(\lim_{x\to\infty} \frac{x+1}{x}\right) = \ln(1) = 0$$

     since the limit of $\frac{x+1}{x}$ as $x \to \infty$ is 1.

   - Find a simple way of writing the function $h(x) = 1/(e^{f(x)} - 1)$.

     Answer: Using the expression (1) for $f(x)$, we see that

$$h(x) = \frac{1}{e^{f(x)} - 1} = \frac{1}{e^{\ln\left(\frac{x+1}{x}\right)} - 1}.$$

Since $e^{\ln(y)} = y$ for any $y > 0$, this simplifies as

$$h(x) = \frac{1}{\frac{x+1}{x} - 1}.$$

Simplifying the denominator yields

$$\frac{x+1}{x} - 1 = \frac{x+1}{x} - \frac{x}{x} = \frac{1}{x},$$

so

$$h(x) = \frac{1}{\frac{x+1}{x} - 1} = \frac{1}{\frac{1}{x}} = x.$$

3. Integrating by parts, compute

$$\int xe^{-x}dx \quad \text{and} \quad \int x^2e^{-x}dx.$$

Show your work. You may use *Mathematica* (or another device or table) to check your answers.

Answer: Consider first the integral

$$\int xe^{-x}dx.$$

Letting $u = x$ and $dv = e^{-x}dx$, we have

$$\begin{aligned} u &= x \quad dv = e^{-x}dx \\ du &= dx \quad v = -e^{-x}. \end{aligned}$$

Therefore, integrating by parts, we have that

$$\int xe^{-x}dx = -xe^{-x} - \int -e^{-x}dx = -xe^{-x} - e^{-x} + C.$$

Now we turn our attention to the integral

$$\int x^2e^{-x}dx.$$

Letting $u = x^2$ and $dv = e^{-x}dx$, we have

$$\begin{aligned} u &= x^2 \quad dv = e^{-x}dx \\ du &= 2xdx \quad v = -e^{-x}. \end{aligned}$$

Therefore, integrating by parts,

$$\int x^2e^{-x}dx = -x^2e^{-x} - \int -2xe^{-x}dx = -x^2e^{-x} + 2\int xe^{-x}dx.$$

We've already figured out $\int xe^{-x}dx$, so just plug that in to get

$$-x^2e^{-x} + 2\left(-xe^{-x} - e^{-x}\right) + C = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C.$$

4. Compute (showing all work)

$$\int_e^{e^2} \frac{1}{x\ln(x)}dx \quad \text{and} \quad \int_0^1 e^x\sqrt{e^x+1}dx.$$

Answer: First, consider the integral

$$\int_e^{e^2} \frac{1}{x\ln(x)}dx.$$

Make the substitution $u = \ln(x)$. Then $du = \frac{1}{x}dx$ and we can re-write the above integral as

$$\int_{u=1}^{2} \frac{1}{u}du$$

(notice that the limits of integration have been converted from x to u). Now we can just compute

$$\begin{aligned}\int_{u=1}^{2} \frac{1}{u}du &= \ln(u)|_{u=1}^{2} \\ &= \ln(2) - \ln(1) \\ &= \ln(2).\end{aligned}$$

Now we turn our attention to the integral

$$\int_0^1 e^x\sqrt{e^x+1}dx.$$

Make the substitution $u = e^x + 1$. Then $du = e^x dx$ and the integral can be re-written as

$$\begin{aligned}\int_{u=2}^{e+1} \sqrt{u}du &= \frac{2}{3}u^{3/2}\Big|_{u=2}^{e+1} \\ &= \frac{2}{3}\left[(e+1)^{3/2} - 2^{3/2}\right] \\ &= \frac{2}{3}\left[(e+1)^{3/2} - 2\sqrt{2}\right]\end{aligned}$$

5. Let $f(x) = \frac{x}{x^2+1}$.

   - Compute $\int f(x)dx$.
     Answer: Make the substitution $u = x^2 + 1$. Then $du = 2xdx$ and the integral can be re-written as

$$\begin{aligned}\frac{1}{2}\int \frac{1}{u}du &= \frac{1}{2}\ln(u) + C \\ &= \frac{1}{2}\ln(x^2+1) + C \\ &= \ln\sqrt{x^2+1} + C.\end{aligned}$$

- Is $\int_0^{100} f(x)dx$ a large number or a small number? Compute it to four decimal places.
  Answer: Using the Fundamental Theorem of Calculus,

  $$\int_0^{100} f(x)dx = \ln\sqrt{100^2+1} - \ln\sqrt{0^2+1} = \ln\sqrt{100^2+1},$$

  since $\ln(1) = 0$. Using a calculator,

  $$\ln\sqrt{100^2+1} \approx 4.6052,$$

  which isn't such a big number.
  Even without using a calculator, we know that

  $$\sqrt{100^2+1} \approx 100,$$

  so

  $$\int_0^{100} f(x)dx \approx \ln(100) \approx 4.6052,$$

  which agrees with the above answer to four decimal places.
- What happens to $\int_0^x f(x)dx$ as $x \to \infty$?
  Answer: Again using the Fundamental Theorem of Calculus,

  $$\int_0^x f(x)dx = \ln\sqrt{x^2+1} - \ln(0^2+1) = \ln\sqrt{x^2+1}.$$

  When x is very large, $\sqrt{x^2+1}$ is very close to x; since $\lim_{x\to\infty}\ln(x) = \infty$, we see that $\int_0^x f(x)dx$ goes to ∞ as $x \to \infty$.

6. *(Extra Credit)* Repeat all parts of Problem #5 with the function $f(x) = \frac{1}{x^2+1}$.

- Compute $\int f(x)dx$.
  Answer: Make the trigonometric substitution $x = \tan\theta$. Then $dx = \sec^2\theta d\theta$ and the integral can be re-written as

  $$\int \frac{1}{\tan^2\theta+1}\sec^2\theta d\theta = \int \frac{1}{\sec^2\theta}\sec^2\theta d\theta = \int d\theta = \theta + C.$$

  Now we need to convert this back to an expression in terms of x. Since $x = \tan\theta$, the quantity $\theta = \tan^{-1} x$, so we have that

  $$\int \frac{1}{x^2+1}dx = \tan^{-1} x + C.$$

- Is $\int_0^{100} f(x)dx$ a large number or a small number? Compute it to four decimal places.
  Answer: Using the Fundamental Theorem of Calculus,

  $$\int_0^{100} f(x)dx = \tan^{-1}(100) - \tan^{-1}(0) = \tan^{-1}(100) \approx 1.5608$$

  using a calculator.

- What happens to $\int_0^x f(x)dx$ as $x \to \infty$?
Answer: Using the Fundamental Theorem of Calculus,

$$\int_0^x f(x)dx = \tan^{-1} x - \tan^{-1}(0) = \tan^{-1} x.$$

Since $\lim_{x\to\pi/2^+} \tan x = +\infty$, we know that

$$\tan^{-1} x \to \pi/2$$

as $x \to \infty$.

7. *(Extra Credit)*

- Compute $\int \frac{x}{(1-x^2)^2} dx$.
Answer: Make the substitution $u = 1 - x^2$. Then $du = -2xdx$ and we can re-write the integral as

$$\begin{aligned}\frac{-1}{2}\int \frac{1}{u^2} du &= \frac{-1}{2}\int u^{-2} du \\ &= \frac{-1}{2}\left[-u^{-1}\right] + C \\ &= \frac{1}{2u} + C \\ &= \frac{1}{2(1-x^2)} + C.\end{aligned}$$

- Compute $\frac{d}{dx}\left(\frac{x^2}{2(1-x^2)}\right)$.
Answer: Using the quotient rule,

$$\begin{aligned}\frac{d}{dx}\left(\frac{x^2}{2(1-x^2)}\right) &= \frac{2x \cdot 2(1-x^2) - x^2(-4x)}{[2(1-x^2)]^2} \\ &= \frac{4x - 4x^3 + 4x^3}{4(1-x^2)^2} \\ &= \frac{4x}{4(1-x^2)^2} \\ &= \frac{x}{(1-x^2)^2}.\end{aligned}$$

- What the heck is going on?
Answer: In the first part we determined that the antiderivatives of $f(x) = \frac{x}{(1-x^2)^2}$ take the form $\frac{1}{2(1-x^2)} + C$. But we saw in the second part that $\frac{x^2}{2(1-x^2)^2}$ is *also* an antiderivative of $\frac{x}{(1-x^2)^2}$.
The only way this can be true is if

$$\frac{x^2}{2(1-x^2)^2} = \frac{1}{2(1-x^2)} + C$$

for some C. In fact, the above equality holds when $C = -1/2$:

$$\frac{1}{2(1-x^2)} - \frac{1}{2} = \frac{1}{2(1-x^2)} - \frac{1-x^2}{2(1-x^2)} = \frac{x^2}{2(1-x^2)}.$$