

## Math 115 HW #11 Solutions

1. Show that the power series solution of the differential equation

$$y' - py = 0$$

is equivalent to the solution found using some other technique.

**Answer:** Suppose  $y = \sum_{k=0}^{\infty} a_k x^k$ . Then

$$y' = \sum_{k=1}^{\infty} k a_k x^{k-1}.$$

Then, since  $y$  is a solution to the given equation,

$$\begin{aligned} 0 &= \sum_{k=1}^{\infty} k a_k x^{k-1} - p \sum_{k=0}^{\infty} a_k x^k \\ 0 &= \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k - \sum_{k=0}^{\infty} p a_k x^k \\ 0 &= \sum_{k=0}^{\infty} [(k+1) a_{k+1} - p a_k] x^k. \end{aligned}$$

Hence, for each  $k$ ,

$$0 = (k+1) a_{k+1} - p a_k \implies a_{k+1} = \frac{p}{k+1} a_k.$$

Thus,

$$\begin{aligned} a_1 &= \frac{p}{1} a_0 = p a_0 \\ a_2 &= \frac{p}{2} a_1 = \frac{p^2}{2} a_0 \\ a_3 &= \frac{p}{3} a_2 = \frac{p^3}{3 \cdot 2} a_0 \\ &\vdots \end{aligned}$$

Therefore,

$$y = \sum_{k=0}^{\infty} a_k x^k = a_0 \sum_{k=0}^{\infty} \frac{p^k}{k!} x^k = a_0 \sum_{k=0}^{\infty} \frac{(px)^k}{k!} = a_0 e^{px}.$$

On the other hand, we could have solved this as a separable differential equation, since the given equation is equivalent to

$$y' = py.$$

Separating variables and integrating, we have that

$$\int \frac{dy}{y} = \int p dx.$$

Hence,

$$\ln |y| = px + C,$$

so

$$|y| = e^{px+C} = e^C e^{px} = Ae^{px}.$$

Allowing  $A$  to be negative lets us drop the absolute value, so

$$y = Ae^{px},$$

agreeing with the solution found using power series.

2. Use power series to solve the differential equation

$$y'' - xy' - y = 0$$

and find the interval of convergence of the series.

**Answer:** Suppose  $y = \sum_{k=0}^{\infty} a_k x^k$ . Then

$$\begin{aligned} y' &= \sum_{k=1}^{\infty} k a_k x^{k-1} \\ y'' &= \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k. \end{aligned}$$

Therefore, since  $y$  is supposed to solve the equation, we should have

$$\begin{aligned} 0 &= y'' - xy' - y \\ 0 &= \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - x \sum_{k=1}^{\infty} k a_k x^{k-1} - \sum_{k=0}^{\infty} a_k x^k \\ 0 &= \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=1}^{\infty} k a_k x^k - \sum_{k=0}^{\infty} a_k x^k \\ 0 &= 2a_2 - a_0 + \sum_{k=1}^{\infty} [(k+2)(k+1) a_{k+2} - k a_k - a_k] x^k. \end{aligned}$$

Therefore,

$$0 = 2a_2 - a_0 \implies a_2 = \frac{1}{2} a_0$$

$$0 = (k+2)(k+1) a_{k+2} - k a_k - a_k \implies a_{k+2} = \frac{k+1}{(k+2)(k+1)} a_k = \frac{1}{k+2} a_k.$$

Therefore,

$$\begin{aligned} a_3 &= \frac{1}{3} a_1 \\ a_4 &= \frac{1}{4} a_2 = \frac{1}{4 \cdot 2} a_0 = \frac{1}{2^2 2!} a_0 \\ a_5 &= \frac{1}{5} a_3 = \frac{1}{5 \cdot 3} a_1 \\ a_6 &= \frac{1}{6} a_4 = \frac{1}{6 \cdot 4 \cdot 2} a_0 = \frac{1}{2^3 3!} a_0: \end{aligned}$$

Therefore,

$$y = \sum_{k=0}^{\infty} a_k x^k = a_0 \sum_{k=0}^{\infty} \frac{x^k}{2^k k!} + a_1 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{1 \cdot 3 \cdot 5 \cdots (2k+1)}.$$

3. Find the first three terms of each of the power series representing independent solutions of the differential equation

$$y'' + x^2 y = 0.$$

**Answer:** Assuming  $y$  can be written as a power series, we can use the expressions already determined for  $y$  and  $y''$  to see that

$$\begin{aligned} 0 &= y'' + x^2 y \\ 0 &= \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k + x^2 \sum_{k=0}^{\infty} a_k x^k \\ 0 &= \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k + \sum_{k=0}^{\infty} a_k x^{k+2} \\ 0 &= \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k + \sum_{k=2}^{\infty} a_{k-2}x^k \\ 0 &= 2a_2 + 6a_3x + \sum_{k=2}^{\infty} [(k+2)(k+1)a_{k+2} + a_{k-2}]x^k \end{aligned}$$

Therefore,

$$\begin{aligned} 0 = 2a_2 &\implies a_2 = 0 \\ 0 = 6a_3 &\implies a_3 = 0 \\ 0 = (k+2)(k+1)a_{k+2} + a_{k-2} &\implies a_{k+2} = -\frac{1}{(k+2)(k+1)}a_{k-2}. \end{aligned}$$

Hence

$$\begin{aligned} a_4 &= -\frac{1}{4 \cdot 3}a_0 \\ a_5 &= -\frac{1}{5 \cdot 4}a_1 \\ a_6 &= -\frac{1}{6 \cdot 5}a_2 = 0 \\ a_7 &= -\frac{1}{7 \cdot 6}a_3 = 0 \\ a_8 &= -\frac{1}{8 \cdot 7}a_4 = \frac{1}{8 \cdot 7 \cdot 4 \cdot 3}a_0 \\ a_9 &= -\frac{1}{9 \cdot 8}a_5 = \frac{1}{9 \cdot 8 \cdot 5 \cdot 4}a_1 \\ a_{10} &= -\frac{1}{10 \cdot 9}a_6 = 0 \\ a_{11} &= -\frac{1}{11 \cdot 10}a_7 = 0 \end{aligned}$$

Hence,

$$y = \sum_{k=0}^{\infty} a_k x^k = a_0 \left( 1 - \frac{x^4}{4 \cdot 3} + \frac{x^8}{8 \cdot 7 \cdot 4 \cdot 3} - \dots \right) + a_1 \left( x - \frac{x^5}{5 \cdot 4} + \frac{x^9}{9 \cdot 8 \cdot 5 \cdot 4} - \dots \right)$$

4. Use Taylor's theorem to solve the initial-value problem

$$y'' - 2xy = 0, \quad y(0) = 1, \quad y'(0) = -3.$$

Use 6 terms of the series to approximate  $y$  at  $x = 1/4$ .

**Answer:** By Taylor's Theorem,

$$y = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \dots$$

Now, we're given  $y(0) = 1$  and  $y'(0) = -3$ , so we just need to figure out the next few terms. The given equation says that

$$y'' = 2xy,$$

so

$$y''(0) = 2 \cdot 0 \cdot y(0) = 2(0)(1) = 0.$$

In turn,

$$y''' = 2y + 2xy',$$

so

$$y'''(0) = 2y(0) + 2 \cdot 0 \cdot y'(0) = 2(1) + 0 = 2.$$

Next,

$$y^{(4)} = 2y' + 2y' + 2xy'' = 4y' + 2xy'',$$

so

$$y^{(4)}(0) = 4y'(0) + 2 \cdot 0 \cdot y''(0) = 4(-3) - 0 = -12.$$

Finally,

$$y^{(5)} = 4y'' + 2y'' + 2xy''' = 6y'' + 2xy''',$$

so

$$y^{(5)}(0) = 6y''(0) + 2 \cdot 0 \cdot y'''(0) = 0 + 0 = 0.$$

Thus, the first 6 terms of the series for  $y$  are

$$y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \frac{y^{(5)}(0)}{5!}x^5 = 1 - 3x + \frac{0}{2!}x^2 + \frac{2}{3!}x^3 - \frac{12}{4!}x^4 + \frac{0}{5!}x^5.$$

This simplifies as

$$1 - 3x + \frac{x^3}{3} - \frac{x^4}{2}.$$

Hence,

$$y(1/4) \approx 1 - 3(1/4) + \frac{(1/4)^3}{3} - \frac{(1/4)^4}{2} = 1 - \frac{3}{4} + \frac{1}{192} - \frac{1}{512} = \frac{389}{1536} \approx 0.253.$$

5. Solve the initial-value problem

$$y'' - 2xy' + 8y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

**Answer:** Assume  $y$  is a power series that solves the given equation. Note, first of all, that

$$y'(0) = \sum_{k=1}^{\infty} k a_k x^{k-1} = 1 \cdot a_1 = a_1;$$

since  $y'(0) = 0$ , this means that  $a_1 = 0$ . Now, since  $y$  solves the given equation,

$$\begin{aligned} 0 &= \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k - 2x \sum_{k=1}^{\infty} k a_k x^{k-1} + 8 \sum_{k=0}^{\infty} a_k x^k \\ 0 &= \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k - \sum_{k=1}^{\infty} 2k a_k x^k + \sum_{k=0}^{\infty} 8a_k x^k \\ 0 &= 2a_2 + 8a_0 + \sum_{k=1}^{\infty} [(k+2)(k+1)a_{k+2} - 2k a_k + 8a_k] x^k \end{aligned}$$

Therefore,

$$\begin{aligned} 0 = 2a_2 + 8a_0 &\implies a_2 = -\frac{1}{4}a_0 \\ 0 = (k+2)(k+1)a_{k+2} - 2k a_k + 8a_k &\implies a_{k+2} = \frac{2k-8}{(k+2)(k+1)}a_k \end{aligned}$$

Therefore,

$$\begin{aligned} a_3 &= \frac{-6}{3 \cdot 2} a_1 = 0 \\ a_4 &= \frac{-4}{4 \cdot 3} a_2 = \frac{1}{4 \cdot 3} a_0 \\ a_5 &= \frac{-2}{5 \cdot 4} a_3 = 0 \\ a_6 &= \frac{0}{6 \cdot 5} a_4 = 0 \\ a_7 &= \frac{2}{7 \cdot 6} a_5 = 0 \\ a_8 &= \frac{4}{8 \cdot 7} a_6 = 0 \\ &\vdots \end{aligned}$$

Thus, we see that

$$y = \sum_{k=0}^{\infty} a_k x^k = a_0 \left( 1 - \frac{x^2}{4} + \frac{x^4}{12} \right).$$

Since  $y(0) = 1$ , we see that

$$1 = a_0(1 - 0 + 0) = a_0,$$

so conclude that

$$y = 1 - \frac{x^2}{4} + \frac{x^4}{12}.$$

6. Solve the initial-value problem

$$y'' - 2xy' + 8y = 0, \quad y(0) = 0, \quad y'(0) = 1$$

(Notice that the differential equation is the same as in #5 above—only the initial conditions are different. This differential equation is known as *Hermite's Equation* of order 4.)

**Answer:** Notice that the recursion relation is the same as in #5 (since it is the same differential equation), but now  $a_1 = y'(0) = 1$ , whereas  $a_0 = y(0) = 0$ . Hence,

$$\begin{aligned} a_2 &= -\frac{1}{4}a_0 = 0 \\ a_3 &= \frac{-6}{3 \cdot 2}a_1 = -1 \\ a_4 &= \frac{-4}{4 \cdot 3}a_2 = 0 \\ a_5 &= \frac{-2}{5 \cdot 4}a_3 = \frac{2}{5 \cdot 4} \\ a_6 &= \frac{0}{6 \cdot 5}a_4 = 0 \\ a_7 &= \frac{2}{7 \cdot 6}a_5 = \frac{2 \cdot 2}{7 \cdot 6 \cdot 5 \cdot 4} \\ a_8 &= \frac{4}{8 \cdot 7}a_6 = 0 \\ a_9 &= \frac{6}{9 \cdot 8}a_7 = \frac{2 \cdot 2 \cdot 6}{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4} \\ &\vdots \end{aligned}$$

Therefore,

$$y = \sum_{k=0}^{\infty} a_k x^k = x - x^3 + \frac{x^5}{10} + 2 \sum_{k=3}^{\infty} \frac{2 \cdot 6 \cdot 10 \cdots (2k-12)}{k!/3!} x^k$$