Math 115 HW #10 Solutions

1. Suppose $y_1(t)$ and $y_2(t)$ are both solutions of the differential equation

$$P(t)y'' + Q(t)y' + R(t)y = 0.$$

Show that, for any constants C_1 and C_2, the function

$$C_1y_1(t) + C_2y_2(t)$$

is also a solution of this differential equation.

Proof. Let $y = C_1y_1 + C_2y_2$. Then

$$y' = C_1y_1' + C_2y_2'$$

and

$$y'' = C_1y_1'' + C_2y_2''.$$

Therefore,

$$\begin{aligned} P(t)y'' + Q(t)y' + R(t)y &= P(t)(C_1y_1'' + C_2y_2'') + Q(t)(C_1y_1' + C_2y_2') + R(t)(C_1y_1 + C_2y_2) \\ &= (P(t)C_1y_1'' + Q(t)C_1y_1' + R(t)C_1y_1) + (P(t)C_2y_2'' + Q(t)C_2y_2' + R(t)C_2y_2) \\ &= C_1(P(t)y_1'' + Q(t)y_1' + R(t)y_1) + C_2(P(t)y_2'' + Q(t)y_2' + R(t)y_2). \end{aligned}$$

However, both terms in the bottom line are zero, for the following reason: since y_1 is a solution of the given differential equation,

$$P(t)y_1'' + Q(t)y_1' + R(t)y_1 = 0;$$

likewise, since y_2 is a solution,

$$P(t)y_2'' + Q(t)y_2' + R(t)y_2 = 0.$$

Therefore, we see that $y = C_1y_1 + C_2y_2$ is indeed a solution of the given differential equation for any constants C_1 and C_2. □

2. Solve the differential equation

$$6y'' - 7y' - 12y = 0.$$

Answer: The characteristic equation is

$$6r^2 - 7r - 12 = 0;$$

solutions of this equation are:

$$r = \frac{7 \pm \sqrt{(-7)^2 - 4(6)(-12)}}{2(6)} = \frac{7 \pm \sqrt{49 + 288}}{12} = \frac{7 \pm \sqrt{337}}{12}.$$

Therefore, solutions of the given differential equation are of the form

$$y = C_1e^{\frac{7+\sqrt{337}}{12}t} + C_2e^{\frac{7-\sqrt{337}}{12}t}.$$

3. Solve the initial-value problem

$$2y'' + 6y' + 17y = 0, \quad y(0) = 1, y'(0) = 5.$$

Answer: The characteristic equation is

$$2r^2 + 6r + 17 = 0;$$

solutions are

$$r = \frac{-6 \pm \sqrt{6^2 - 4(2)(17)}}{2(2)} = \frac{-6 \pm \sqrt{36 - 136}}{4} = -\frac{3}{2} \pm \frac{5}{2}i.$$

Therefore, solutions of the given differential equation are of the form

$$y = C_1 e^{-3/2t} \cos\left(\frac{5}{2}t\right) + C_2 e^{-3/2t} \sin\left(\frac{5}{2}t\right).$$

Plugging in $t = 0$, we have that

$$1 = y(0) = C_1 e^0 \cos(0) + C_2 e^0 \sin(0) = C_1,$$

so $C_1 = 1$ and

$$y = e^{-3/2t} \cos\left(\frac{5}{2}t\right) + C_2 e^{-3/2t} \sin\left(\frac{5}{2}t\right).$$

Hence,

$$y' = -\frac{3}{2}e^{-3/2t} \cos\left(\frac{5}{2}t\right) - \frac{5}{2}e^{-3/2t} \sin\left(\frac{5}{2}t\right) - \frac{3}{2}C_2 e^{-3/2t} \sin\left(\frac{5}{2}t\right) + \frac{5}{2}C_2 e^{-3/2t} \cos\left(\frac{5}{2}t\right)$$
$$y' = \left(\frac{5}{2}C_2 - \frac{3}{2}\right) e^{-3/2t} \cos\left(\frac{5}{2}t\right) - \left(\frac{5}{2} + \frac{3}{2}C_2\right) e^{-3/2t} \sin\left(\frac{5}{2}t\right).$$

Therefore, plugging in $t = 0$ yields

$$5 = y'(0) = \left(\frac{5}{2}C_2 - \frac{3}{2}\right) e^0 \cos(0) - \left(\frac{5}{2} + \frac{3}{2}C_2\right) e^0 \sin(0) = \frac{5}{2}C_2 - \frac{3}{2}.$$

Therefore,

$$\frac{5}{2}C_2 = 5 + \frac{3}{2} = \frac{13}{2},$$

so

$$C_2 = \frac{2}{5}\frac{13}{2} = \frac{13}{5}.$$

Thus, we conclude that

$$y = e^{-3/2t} \cos\left(\frac{5}{2}t\right) + \frac{13}{5}e^{-3/2t} \sin\left(\frac{5}{2}t\right).$$

4. A spring-mass-dashpot system (like the door-closing mechanism in many doors) can be modeled by the differential equation

$$mx'' + cx' + kx = 0$$

where x is the displacement of the object, m is the mass of the object, c is the damping constant for the dashpot, and k is the spring constant. Suppose we have such a system with a mass $m = 20$ kg, a spring with $k = 5$, and a dashpot whose damping constant c we can adjust. What value of c should we pick to get *critical damping*?

Answer: Critical damping occurs when there is a single root (of multiplicity 2) of the characteristic equation. Plugging in the values for m and k we have the differential equation

$$20x'' + cx' + 5x = 0,$$

so the characteristic equation is

$$20r^2 + cr + 5 = 0.$$

Solutions are of the form

$$r = \frac{-c \pm \sqrt{c^2 - 4(20)(5)}}{2(20)} = \frac{-c \pm \sqrt{c^2 - 400}}{40}.$$

There is a single root of multiplicity 2 when the discriminant $c^2 - 400 = 0$, meaning that

$$c = \pm 20.$$

The value $c = -20$ is physically meaningless, so we should pick $c = 20$ to get critical damping.

5. Solve the differential equation

$$y'' - y' - 6y = e^{2x}.$$

Answer: First, we solve the homogeneous equation

$$y'' - y' - 6y = 0.$$

This has characteristic equation

$$r^2 - r - 6 = 0;$$

The left side factors as $(r - 3)(r + 2)$, so the roots are $r_1 = 3$ and $r_2 = -2$. Hence, the complementary solution (i.e. solution to the homogeneous equation) is

$$y_c = C_1 e^{3x} + C_2 e^{-2x}.$$

Now, we need to find a particular solutions to the given equation

$$y'' - y' - 6y = e^{2x}.$$

Using the method of undetermined coefficients, guess that

$$y_p = Ae^{2x}.$$

Then
$$y_p' = 2Ae^{2x}$$
and
$$y_p'' = 4Ae^{2x}.$$
Therefore, if y_p really is a solution, we should have that
$$e^{2x} = y_p'' - y_p' - 6y_p = 4Ae^{2x} - 2Ae^{2x} - 6(Ae^{2x}) = -4Ae^{2x}.$$
Therefore, it must be the case that
$$1 = -4A,$$
so $A = -\frac{1}{4}$ and
$$y_p = -\frac{1}{4}e^{2x}.$$
Thus, the general solution of the given non-homogeneous equation is
$$y = y_c + y_p = C_1e^{3x} + C_2e^{-2x} - \frac{1}{4}e^{2x}.$$

6. Solve the differential equation
$$y'' - 4y' + 4y = e^{2x}.$$

Answer: First, solve the homogeneous equation
$$y'' - 4y' + 4y = 0.$$
This has characteristic equation
$$r^2 - 4r + 4 = 0$$
and the left side factors as $(r-2)^2$, so the single solution (of multiplicity 2) is $r = 2$. Therefore, the complementary solution is
$$y_c = C_1e^{2x} + C_2xe^{2x}.$$
Now, to find a particular solution to the given equation, we would like to guess that y_p is e^{2x}. However, this is already a solution to the homogeneous equation, so it can't be a particular solution. Multiplying by x yields xe^{2x}, which is also a solution to the homogeneous equation. Therefore, we need to multiply by x again and guess
$$y_p = Ax^2e^{2x}.$$
Then
$$y_p' = 2Axe^{2x} + 2Ax^2e^{2x}$$
and
$$\begin{aligned} y_p'' &= 2Ae^{2x} + 4Axe^{2x} + 4Axe^{2x} + 4Ax^2e^{2x} \\ &= 2Ae^{2x} + 8Axe^{2x} + 4Ax^2e^{2x}. \end{aligned}$$

Therefore, since y_p is a solution of the equation,

$$\begin{aligned}
e^{2x} &= y_p'' - 4y_p' + 4y_p \\
&= (2Ae^{2x} + 8Axe^{2x} + 4Ax^2e^{2x}) - 4(2Axe^{2x} + 2Ax^2e^{2x}) + 4Ax^2e^{2x} \\
&= (4A - 8A + 4A)x^2e^{2x} + (8A - 8A)xe^{2x} + 2Ae^{2x} \\
&= 2Ae^{2x}.
\end{aligned}$$

Therefore, $2A = 1$ and so $A = 1/2$. Hence,

$$y_p = \frac{1}{2}x^2e^{2x}$$

and so the solution of the differential equation is

$$y = y_c + y_p = C_1e^{2x} + C_2xe^{2x} + \frac{1}{2}x^2e^{2x}.$$

7. Solve the initial-value problem

$$y'' + 9y = \cos 3x + \sin 3x, \quad y(0) = 2, y'(0) = 1.$$

Answer: First, solve the homogeneous equation

$$y'' + 9y = 0.$$

This equation has characteristic equation

$$r^2 + 9 = 0,$$

which has solutions $r = \pm 3i$. Therefore, the complementary solution is

$$\begin{aligned}
y_c &= C_1e^{0x}\cos 3x + C_2e^{0x}\sin 3x \\
&= C_1\cos 3x + C_2\sin 3x.
\end{aligned}$$

Now, we would like to guess that the particular solution is $A\cos 3x + B\sin 3x$, but both $\cos 3x$ and $\sin 3x$ are solutions to the homogeneous equation. Hence, we multiply by x and guess that

$$y_p = Ax\cos 3x + Bx\sin 3x.$$

Then

$$\begin{aligned}
y_p' &= A\cos 3x - 3Ax\sin 3x + B\sin 3x + 3Bx\cos 3x \\
&= (A + 3Bx)\cos 3x + (B - 3Ax)\sin 3x
\end{aligned}$$

and so

$$\begin{aligned}
y_p'' &= -3A\sin 3x + 3B\cos 3x - 9Bx\sin 3x + 3B\cos 3x - 3A\sin 3x - 9Ax\cos 3x \\
&= (6B - 9Ax)\cos 3x - (6A + 9Bx)\sin 3x.
\end{aligned}$$

Therefore, since y_p solves the differential equation,

$$\begin{aligned}\cos 3x + \sin 3x &= y_p'' + 9y_p \\ &= [(6B - 9Ax)\cos 3x - (6A + 9Bx)\sin 3x] + 9\,[Ax\cos 3x + Bx\sin 3x] \\ &= 6B\cos 3x - 6A\sin 3x\end{aligned}$$

Therefore

$$1 = 6B, \qquad 1 = -6A,$$

so $B = 1/6$ and $A = -1/6$, meaning that

$$y_p = -\frac{1}{6}x\cos 3x + \frac{1}{6}x\sin 3x.$$

Hence,

$$y = y_c + y_p = C_1\cos 3x + C_2\sin 3x - \frac{1}{6}x\cos 3x + \frac{1}{6}x\sin 3x.$$

Now, plugging in $x = 0$ yields

$$2 = y(0) = C_1\cos(0) + C_2\sin(0) - \frac{1}{6}(0)\cos(0) + \frac{1}{6}(0)\sin(0) = C_1,$$

so $C_1 = 2$ and

$$y = 2\cos 3x + C_2\sin 3x - \frac{1}{6}x\cos 3x + \frac{1}{6}x\sin 3x.$$

Our other initial value is $y'(0) = 1$, so we need to find the derivative of y:

$$\begin{aligned}y' &= -6\sin 3x + 3C_2\cos 3x - \frac{1}{6}\cos 3x + \frac{1}{2}x\sin 3x + \frac{1}{6}\sin 3x + \frac{1}{2}x\cos 3x \\ &= \left(3C_2 - \frac{1}{6} + \frac{1}{2}x\right)\cos 3x + \left(-6 + \frac{1}{6} + \frac{1}{2}x\right)\sin 3x.\end{aligned}$$

Plugging in $x = 0$ yields

$$\begin{aligned}1 = y'(0) &= \left(3C_2 - \frac{1}{6} + \frac{1}{2}(0)\right)\cos(0) + \left(-6 + \frac{1}{6} + \frac{1}{2}(0)\right)\sin(0) \\ &= 3C_2 - \frac{1}{6}.\end{aligned}$$

Hence,

$$3C_2 = \frac{7}{6}$$

so $C_2 = \frac{7}{18}$.

Therefore, finally, we see that

$$y = 2\cos 3x + \frac{7}{18}\sin 3x - \frac{x}{6}\cos 3x + \frac{x}{6}\sin 3x.$$