Math 113 HW #8 Solutions

1. Exercise 3.8.10. A sample of tritium-3 decayed to 94.5% of its original amount after a year.

   (a) What is the half-life of tritium-3?

   Answer: If $N(t)$ is the amount of tritium-3 relative to the original amount, we know that the general form of $N(t)$ is
   $$N(t) = Ce^{kt}.$$
   Also, we know that $N(0) = 1$, so
   $$1 = N(0) = Ce^{k\cdot 0} = C,$$
   so $C = 1$ and we can write
   $$N(t) = e^{kt}.$$
   Also, we know $N(1) = 0.945$, so
   $$0.945 = N(1) = e^{k\cdot 1} = e^k.$$
   Taking the natural log of both sides,
   $$k = \ln(0.945).$$
   Therefore,
   $$N(t) = e^{\ln(0.945)t} = \left(e^{\ln(0.945)}\right)^t = (0.945)^t$$
   for any t.

   The half-life of tritium-3 is the amount of time t_0 such that $N(t_0) = 0.5$. Therefore, we can solve for t_0 from the equation
   $$0.5 = N(t_0) = (0.945)^{t_0}.$$
   Taking the natural log of both sides,
   $$\ln(0.5)\ln(0.945^{t_0}) = t_0 \ln(0.945),$$
   so
   $$t_0 = \frac{\ln(0.5)}{\ln(0.945)} \approx 12.25.$$
   Therefore, the half-life of tritium-3 is 12.25 years.

   (b) How long would it take the sample to decay to 20% of its original amount?

   Answer: If t_1 is the time it takes the sample to decay to 20% of its original amount,
   $$0.2 = N(t_1) = (0.945)^{t_1},$$
   meaning that (if we take the natural log of both sides),
   $$\ln(0.2) = \ln(0.945^{t_1}) = t_1 \ln(0.945),$$
   so
   $$t_1 = \frac{\ln(0.2)}{\ln(0.945)} \approx 28.45 \text{ years}.$$

2. Exercise 3.8.14. A thermometer is taken from a room where the temperature is 20°C to the outdoors, where the temperature is 5°C. After one minute the thermometer reads 12°C.

   (a) What will the reading on the thermometer be after one more minute?
   Answer: From Newton's Law of Cooling, we know that

$$T(t) = T_s + Ce^{kt}.$$

The ambient temperature is $T_s = 5$°C, whereas

$$20 = T(0) = 5 + Ce^{k\cdot 0} = 5 + C,$$

so $C = 15$. Therefore,

$$T(t) = 5 + 15e^{kt}.$$

Moreover, we know that $T(1) = 12$, so

$$12 = T(1) = 5 + 15e^{k\cdot 1} = 5 + 15e^{k},$$

so

$$e^k = \frac{7}{15}.$$

Taking the natural log of both sides,

$$k = \ln\left(\frac{7}{15}\right).$$

Hence,

$$T(t) = 5 + 15e^{\ln\left(\frac{7}{15}\right)t} = 5 + 15\left(e^{\ln\left(\frac{7}{15}\right)}\right)^t = 5 + \left(\frac{7}{15}\right)^t.$$

Therefore, after 2 minutes, the temperature of the thermometer will be

$$T(2) = 5 + 15\left(\frac{7}{15}\right)^2 = 5 + \frac{49}{15} = \frac{124}{15} = 8.266\ldots$$

   (b) When will the thermometer read 6°C?
   Answer: The time t_0 when $T(t_0) = 6$ is given by

$$6 = T(t_0) = 5 + 15\left(\frac{7}{15}\right)^{t_0},$$

so

$$\frac{1}{15} = \left(\frac{7}{15}\right)^{t_0}.$$

Taking the natural log of both sides,

$$\ln\left(\frac{1}{15}\right) = \ln\left(\frac{7}{15}\right)^{t_0} = t_0 \ln\left(\frac{7}{15}\right).$$

Therefore,

$$t_0 = \frac{\ln\frac{1}{15}}{\ln\frac{7}{15}} \approx 3.55,$$

so the thermometer will read 6°C after about 3 and a half minutes.

3. Exercise 3.8.16. A freshly brewed cup of coffee has temperature 95°C in a 20°C room. When its temperature is 70°C, it is cooling at a rate of 1°C per minute. When does this occur?

   Answer: From Newton's Law of Cooling, the temperature of the coffee is given by

$$T(t) = 20 + Ce^{kt}.$$

   At time $t = 0$,

$$95 = T(0) = 20 + Ce^{k\cdot 0} = 20 + C,$$

   meaning that $C = 75$ and $T(t) = 20 + 75e^{kt}$. At some time t_0,

$$70 = T(t_0) = 20 + 75e^{kt_0},$$

   so

$$75e^{kt_0} = 50,$$

   or

$$e^{kt_0} = \frac{2}{3}.$$

   Therefore,

$$kt_0 = \ln\left(\frac{2}{3}\right).$$

   We also know the rate of change of T at this time t_0:

$$-1 = T'(t_0) = 75(e^{kt_0}k) = 75ke^{kt_0}.$$

   In other words,

$$k = \frac{-1}{75e^{kt_0}}.$$

   Since $kt_0 = \ln\left(\frac{2}{3}\right)$, we know that

$$k = \frac{-1}{75e^{\ln\left(\frac{2}{3}\right)}} = \frac{-1}{75\frac{2}{3}} = \frac{-1}{50}.$$

   Since $kt_0 = \ln\left(\frac{2}{3}\right)$, we know that

$$t_0 = \frac{\ln\left(\frac{2}{3}\right)}{k} = \frac{\ln\left(\frac{2}{3}\right)}{\frac{-1}{50}} = -50\ln\left(\frac{2}{3}\right) \approx 20.3,$$

   so the cup of coffee is 70°C after just over 20 minutes.

4. Exercise 3.10.12. Find the differential of the functions

   (a) $y = s/(1+2s)$

   Answer: If $f(s) = \frac{s}{1+2s}$, then, by definition,

$$dy = f'(s)ds.$$

   Now,

$$f'(s) = \frac{(1+2s)\cdot 1 - s\cdot 2}{(1+2s)^2} = \frac{1+2s-2s}{(1+2s)^2} = \frac{1}{(1+2s)^2}.$$

   Therefore, the differential is

$$dy = \frac{ds}{(1+2s)^2}.$$

(b) $y = e^{-u} \cos u$

Answer: If $g(u) = e^{-u} \cos u$, then, by definition,

$$dy = g'(u)du.$$

Since

$$g'(u) = -e^{-u} \cos u + e^{-u}(-\sin u) = -e^{-u} \cos u - e^{-u} \sin u = -e^{-u}(\cos u + \sin u),$$

we have that

$$dy = -e^{-u}(\cos u + \sin u)du.$$

5. Exercise 3.10.18.

(a) Find the differential dy of $y = \cos x$.

Answer: By definition, if $f(x) = \cos x$, then

$$dy = f'(x)dx.$$

Since $f'(x) = -\sin x$, this means that

$$dy = -\sin x dx.$$

(b) Evaluate dy for $x = \pi/3$ and $dx = 0.05$.

Answer: Given the above expression for dy and knowing that $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$, we have that

$$dy = -\frac{\sqrt{3}}{2}(0.05) = -\frac{\sqrt{3}}{40} \approx 0.0433.$$

6. Exercise 3.10.24. Use a linear approximation (or differentials) to estimate $e^{-0.015}$.

Answer: Let $f(x) = e^x$. If $L(x)$ is the linearization of f at 0, then

$$L(x) = f(0) + f'(0)(x - 0) = 1 + 1(x - 0) = 1 + x.$$

Since -0.015 is close to 0, it should be the case that

$$e^{-0.015} \approx L(-0.015) = 1 + (-0.015) = 0.985.$$

7. Exercise 3.10.32. Let $f(x) = (x-1)^2$, $g(x) = e^{-2x}$, $h(x) = 1 + \ln(1 - 2x)$.

(a) Find the linearizations of f, g, and h at $a = 0$. What do you notice? How do you explain what happened?

Answer: By definition, the linearization of f is

$$f(0) + f'(0)(x - 0) = f(0) + f'(0)x.$$

Since $f'(x) = 2(x - 1)$, we know that $f(0) = 1$ and $f'(0) = -2$, so the linearization of f is

$$1 - 2x.$$

By definition, the linearization of g is

$$g(0) + g'(0)(x - 0) = g(0) + g'(0)x.$$

Since $g'(x) = -2e^{-2x}$, we know that $g(0) = 1$ and $g'(0) = -2$, so the linearization of g is

$$1 - 2x$$

By definition, the linearization of h is

$$h(0) + h'(0)(x - 0) = h(0) + h'(0)x.$$

Since $h'(x) = \frac{-2}{1-2x}$, we know that $h(0) = 1$ and $h'(0) = -2$, so the linearization of h is

$$1 - 2x.$$

We notice that all three linearizations are the same. This occurs because $f(0) = g(0) = h(0)$ and $f'(0) = g'(0) = h'(0)$: all three functions have the same value at 0 and their derivatives also have the same value at 0. Of course, this says nothing about the behavior of the three functions at other points.

(b) Graph f, g, and h and their linear approximations. For which function is the linear approximation best? For which is it worst? Explain.

Answer:

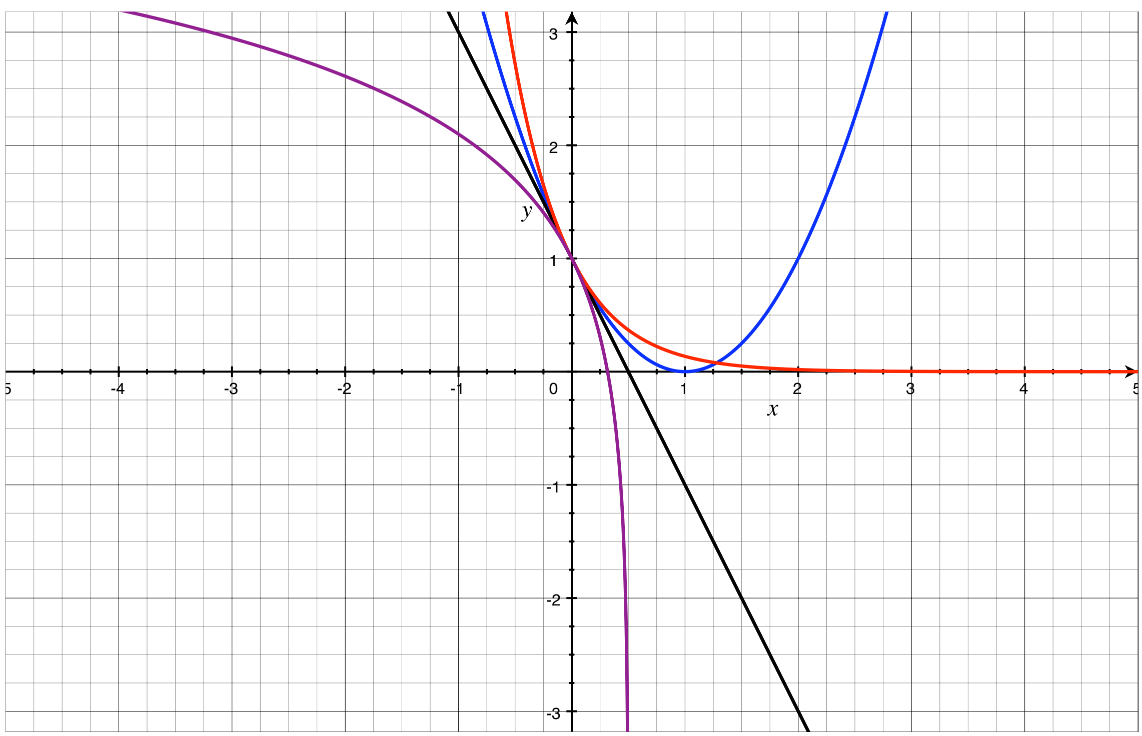

Figure 1: Blue: f; Red: g; Purple: h; Black: linearization

From the picture, the linear approximation appears to be best for f and worst for h.