Math 113 HW #7 Solutions

1. Exercise 3.5.10. Given

$$y^5 + x^2y^3 = 1 + ye^{x^2}$$

find dy/dx by implicit differentiation.

Answer: Differentiating both sides with respect to x yields

$$5y^4\frac{dy}{dx} + 2xy^3 + x^2(3y^2)\frac{dy}{dx} = \frac{dy}{dx}e^{x^2} + y(2x)e^{x^2}.$$

Re-arranging so that all terms containing dy/dx are on the left side gives

$$\left(5y^4 + 3x^2y^2 - e^{x^2}\right)\frac{dy}{dx} = 2xye^{x^2} - 2xy^3.$$

Thus,

$$\frac{dy}{dx} = \frac{2xye^{x^2} - 2xy^3}{5y^4 + 3x^2y^2 - e^{x^2}}.$$

2. Exercise 3.5.12. Given

$$1 + x = \sin(xy^2)$$

find dy/dx by implicit differentiation.

Answer: Differentiating both sides with respect to x yields

$$1 = \cos(xy^2)\left(y^2 + x(2y)\frac{dy}{dx}\right).$$

Therefore,

$$1 - y^2\cos(xy^2) = 2xy\cos(xy^2)\frac{dy}{dx},$$

so we have that

$$\frac{dy}{dx} = \frac{1 - y^2\cos(xy^2)}{2xy\cos(xy^2)}.$$

3. Exercise 3.5.30. Use implicit differentiation to find an equation of the tangent line to the curve

$$y^2(y^2 - 4) = x^2(x^2 - 5)$$

at the point $(0, -2)$.

Answer: Distributing the products, the given equation is equivalent to

$$y^4 - 4y^2 = x^4 - 5x^2.$$

Differentiating with respect to x on both sides, we have that

$$4y^3\frac{dy}{dx} - 8y\frac{dy}{dx} = 4x^3 - 10x.$$

Therefore, factoring out the dy/dx on the left and dividing both sides by $4y^3 - 8y$ yields

$$\frac{dy}{dx} = \frac{4x^3 - 10x}{4y^3 - 8y}.$$

Plugging in $x = 0$, $y = -2$, we see that, at the point $(0, -2)$, the slope of the tangent line is

$$\frac{4(0)^3 - 10(0)}{4(-2)^3 - 8(-2)} = 0.$$

Using the point-slope formula, then, the equation of the tangent line is

$$y - (-2) = 0(x - 0)$$

or, equivalently,

$$y = -2.$$

4. Exercise 3.5.46. Find the derivative of the function

$$y = \sqrt{\tan^{-1} x}.$$

Simplify where possible.

Answer: Writing y as $\left(\tan^{-1} x\right)^{1/2}$ and using the Chain Rule and what we know about the derivative of $\tan^{-1} x$,

$$\frac{dy}{dx} = \frac{1}{2}\left(\tan^{-1} x\right)^{-1/2} \frac{1}{1 + x^2} = \frac{1}{2(1 + x^2)\sqrt{\tan^{-1} x}}.$$

5. Exercise 3.5.60. Two curves are **orthogonal** if their tangent lines are perpendicular at each point of intersection. Show that the two families of curves

$$x^2 + y^2 = ax, \qquad x^2 + y^2 = by$$

are **orthogonal trajectories** of each other, that is, every curve in one family is orthogonal to every curve in the other family. Sketch both families of curves on the same axes.

Answer: Differentiating both sides of the equation for the first family, we see that

$$2x + 2y\frac{dy}{dx} = a,$$

meaning that

$$\frac{dy}{dx} = \frac{a - 2x}{2y} \tag{1}$$

gives the slopes of the tangent lines to the first family of curves.

Turning attention to the second family, differentiating yields

$$2x + 2y\frac{dy}{dx} = b\frac{dy}{dx},$$

meaning that

$$(2y - b)\,\frac{dy}{dx} = -2x,$$

so

$$\frac{dy}{dx} = \frac{-2x}{2y - b} \tag{2}$$

gives the slopes of the tangent lines to the second family of curves.

At a point of intersection (x_0, y_0), the two curves are orthogonal if and only if the expressions (1) and (2) are negative reciprocals, i.e. if and only if

$$\frac{a - 2x_0}{2y_0} = -\frac{2y_0 - b}{-2x_0}.$$

Cross-multiplying, this holds if and only if

$$-2ax_0 + 4x_0^2 = -4y_0^2 + 2by_0$$

which is equivalent to

$$4(x_0^2 + y_0^2) = 2ax_0 + 2by_0 = 2(ax_0 + by_0).$$

But, since $ax_0 = x_0^2 + y_0^2$ and $by_0 = x_0^2 + y_0^2$, this is true, so the two families of curves are indeed orthogonal.

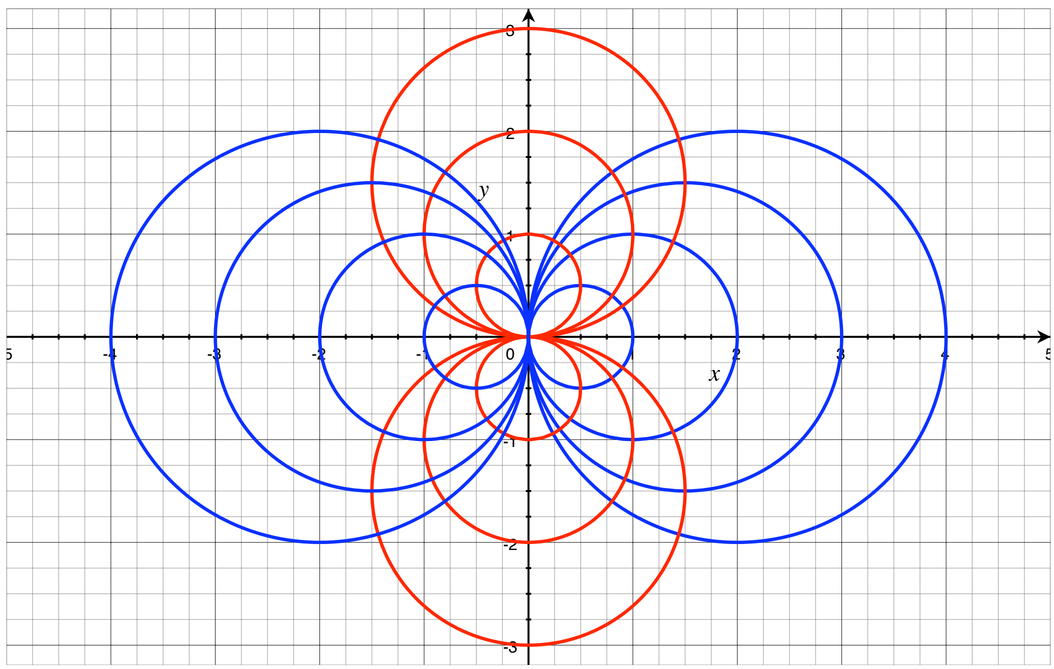

Figure 1: Blue curves: $x^2 + y^2 = ax$; red curves: $x^2 + y^2 = by$

6. Exercise 3.6.16. Differentiate the function

$$y = \frac{1}{\ln x}.$$

Answer: Using the Chain Rule and the fact that $\frac{d}{dx}(\ln x) = \frac{1}{x}$, we see that

$$\frac{dy}{dx} = -(\ln x)^{-2} \cdot \frac{1}{x} = -\frac{1}{x(\ln x)^2}.$$

7. Exercise 3.6.26. Let

$$y = \ln(\sec x + \tan x).$$

Find y' and y''.

Answer: Using the Chain Rule,

$$y' = \frac{1}{\sec x + \tan x}\left(\sec x \tan x + \sec^2 x\right) = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}.$$

Factoring out $\sec x$ from both terms in the numerator, we see that

$$y' = \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} = \sec x.$$

Therefore,

$$y'' = \sec x \tan x.$$

8. Exercise 3.6.40. Use logarithmic differentiation to find the derivative of the function

$$y = \sqrt[4]{\frac{x^2+1}{x^2-1}}.$$

Answer: First, take the natural log of both sides:

$$\ln y = \ln \sqrt[4]{\frac{x^2+1}{x^2-1}} = \ln\left(\frac{x^2+1}{x^2-1}\right)^{1/4} = \frac{1}{4}\ln\left(\frac{x^2+1}{x^2-1}\right).$$

We can further simplify this, using the rules of logarithms, as

$$\ln y = \frac{1}{4}\left(\ln(x^2+1) - \ln(x^2-1)\right).$$

Now, differentiating both sides yields

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{4}\left(\frac{1}{x^2+1}\cdot 2x - \frac{1}{x^2-1}\cdot 2x\right),$$

or, after simplifying,

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{4}\left(\frac{2x}{x^2+1} - \frac{2x}{x^2-1}\right).$$

Multiplying both sides by y gives

$$\frac{dy}{dx} = \frac{1}{4}y\left(\frac{2x}{x^2+1} - \frac{2x}{x^2-1}\right).$$

Finally, plugging in $y = \sqrt[4]{\frac{x^2+1}{x^2-1}}$ yields

$$\frac{dy}{dx} = \frac{1}{4}\sqrt[4]{\frac{x^2+1}{x^2-1}}\left(\frac{2x}{x^2+1} - \frac{2x}{x^2-1}\right).$$

9. Exercise 3.6.42. Use logarithmic differentiation to find the derivative of the function

$$y = x^{\cos x}.$$

Answer: Taking the natural log of both sides,

$$\ln y = \ln\left(x^{\cos x}\right) = \cos x \ln x.$$

Therefore, differentiating yields

$$\frac{1}{y}\frac{dy}{dx} = -\sin x \ln x + \cos x \frac{1}{x} = \frac{\cos x}{x} - \sin x \ln x.$$

Multiplying both sides by y,

$$\frac{dy}{dx} = y\left(\frac{\cos x}{x} - \sin x \ln x\right)$$

or, after plugging in $y = x^{\cos x}$,

$$\frac{dy}{dx} = x^{\cos x}\left(\frac{\cos x}{x} - \sin x \ln x\right).$$

10. Exercise 3.9.12. If a snowball melts so that its surface area decreases at a rate of $1\text{cm}^2/\text{min}$, find the rate at which the diameter decreases when the diameter is 10 cm.

Answer: We know that the surface area of a (spherical) snowball is given by

$$A = 4\pi r^2$$

and that the diameter is $2r$. Also, we know that $\frac{dA}{dt} = -1$ and that, at the time of interest, the diameter is 10 cm, meaning that $r = 5$.

Our goal is to determine the rate at which the diameter decreases, which will be $2\frac{dr}{dt}$.

Differentiating the expression for A yields

$$\frac{dA}{dt} = 4\pi(2r)\frac{dr}{dt} = 8\pi r\frac{dr}{dt}.$$

Therefore, at the time of interest, we can plug in the known values for $\frac{dA}{dt}$ and r:

$$-1 = 8\pi(5)\frac{dr}{dt} = 40\pi\frac{dr}{dt}.$$

Therefore,

$$\frac{dr}{dt} = -\frac{1}{40\pi},$$

so the diameter is decreasing at a rate of $\frac{1}{20\pi} \approx 0.016$ cm/min.

11. Exercise 3.9.32. When air expands adiabatically (without gaining or losing heat), its pressure P and volume V are related by the equation $PV^{1.4} = C$, where C is a constant. Suppose that at a certain instant the volume is 400 cm^3 and the pressure is 80 kPa and is decreasing at a rate of 10 kPa/min. At what rate is the volume increasing at this instant?

Answer: We know that temperature and pressure are related by the equation

$$PV^{1.4} = C.$$

We also know that, at a certain instant,

$$V = 400, \quad P = 80, \quad \frac{dP}{dt} = -10.$$

Our goal is to determine $\frac{dV}{dt}$, so we differentiate the equation relating P and V:

$$\frac{dP}{dt}V^{1.4} + P(1.4)V^{0.4}\frac{dV}{dt} = 0.$$

Solving for $\frac{dV}{dt}$, we see that

$$\frac{dV}{dt} = \frac{-\frac{dP}{dt}V^{1.4}}{1.4PV^{0.4}} = -\frac{\frac{dP}{dt}V}{1.4P}.$$

Therefore, at the instant in question,

$$\frac{dV}{dt} = -\frac{-10 \cdot 400}{1.4 \cdot 80} = 50 \cdot \frac{5}{7} = \frac{250}{7} \approx 35.7,$$

so the volume is increasing at 35.7 cm^3/min.