Math 113 HW #5 Solutions

1. Exercise 2.5.46. Suppose f is continuous on $[1, 5]$ and the only solutions of the equation $f(x) = 6$ are $x = 1$ and $x = 4$. If $f(2) = 8$, explain why $f(3) > 6$.

   Answer: Suppose we had that $f(3) \leq 6$. Then either $f(3) = 6$ or $f(3) < 6$. The first is clearly impossible, because we know $f(x) = 6$ only when $x = 1, 4$.

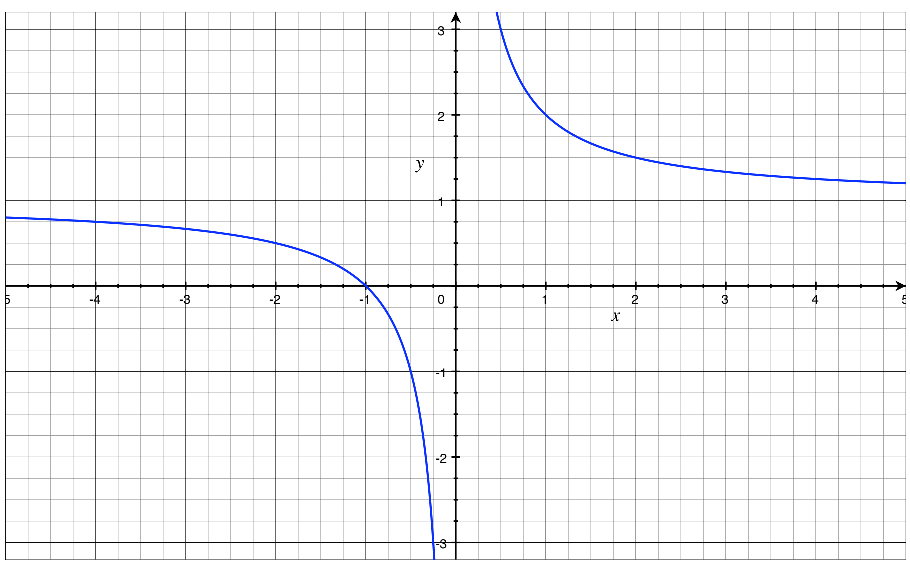
   On the other hand, if $f(3) < 6$, then we have that f is continuous on $[2, 3]$ and $f(2) = 8 > 6 > f(3)$. Therefore, by the Intermediate Value Theorem, there exists a number c between 2 and 3 such that $f(c) = 6$. However, this is again impossible because c is between 2 and 3 and so cannot be equal to 1 or 4.

   Hence, since $f(3) \leq 6$ is impossible, we have to conclude that $f(3) > 6$, as desired.

2. Exercise 2.6.6. Sketch the graph of an example of a function f that satisfies all of the conditions

$$\lim_{x\to 0^+} f(x) = \infty, \quad \lim_{x\to 0^-} f(x) = -\infty, \quad \lim_{x\to\infty} f(x) = 1, \quad \lim_{x\to-\infty} f(x) = 1.$$

   Answer:



3. Exercise 2.6.18. Find the limit

$$\lim_{y\to\infty} \frac{2 - 3y^2}{5y^2 + 4y}$$

   Answer: Dividing both numerator and denominator by y^2, we get

$$\lim_{y\to\infty} \frac{\frac{1}{y^2}\left(2 - 3y^2\right)}{\frac{1}{y^2}\left(5y^2 + 4y\right)} = \lim_{y\to\infty} \frac{\frac{2}{y^2} - 3}{5 + \frac{4}{y}},$$

which, using the Limit Law for quotients, is equal to

$$\frac{\lim_{y\to\infty}\left(\frac{2}{y^2}-3\right)}{\lim_{y\to\infty}\left(5+\frac{4}{y}\right)} = -\frac{3}{5}.$$

Therefore,

$$\lim_{y\to\infty}\frac{2-3y^2}{5y^2+4y} = -\frac{3}{5}.$$

4. Exercise 2.6.28. Find the limit

$$\lim_{x\to\infty}\cos x.$$

Answer: Because $\cos(2n\pi) = 1$ for any n and because $\cos((2n+1)\pi) = -1$ for any n, this limit cannot exist: no matter how far out we go on the x-axis, $\cos x$ is still oscillating between 1 and -1, so it never settles down to a limit.

5. Exercise 2.6.58.

   (a) A tank contains 5000 L of pure water. Brine that contains 30 g of salt per liter of water is pumped into the tank at a rate of 25 L/min. Show that the concentration of salt after t minutes (in grams per liter) is

   $$C(t) = \frac{30t}{200+t}.$$

   Answer: Since the tank starts with 5000 L of water and since 25 L/min of brine pours in, the number of liters of water in the tank is given by

   $$5000 + 25t.$$

   On the other hand, the tank starts with no salt in it and, for each liter of brine that pours in, 30 g of salt pours in. Hence, the number of grams of salt in the tank is given by

   $$25(30t).$$

   Therefore, the concentration of salt is given by

   $$C(t) = \frac{25(30t)}{5000+25t} = \frac{25(30t)}{25(200+t)} = \frac{30t}{200+t}.$$

   (b) What happens to the concentration as $t\to\infty$?
   Answer: As $t\to\infty$, the concentration of salt is given by

   $$\lim_{t\to\infty} C(t) = \lim_{t\to\infty}\frac{30t}{200+t}.$$

   Dividing both numerator and denominator by t yields

   $$\lim_{t\to\infty}\frac{\frac{1}{t}30t}{\frac{1}{t}(200+t)} = \lim_{t\to\infty}\frac{30}{\frac{200}{t}+1} = 30.$$

   So eventually the concentration of salt in the tank approaches 30 g/L, which is the same as the concentration of salt in the brine. In other words, the brine eventually overpowers the pure water.

6. Exercise 2.7.12. Shown are graphs of the position functions of two runners, A and B, who run a 100-m race and finish in a tie.

   (a) Describe and compare how the runners run the race

   Answer: A runs the race at a constant speed, never speeding up or slowing down. B accelerates throughout the race, starting out slower than A and, by the end, running faster than A.

   (b) At what time is the distance between the runners the greatest?

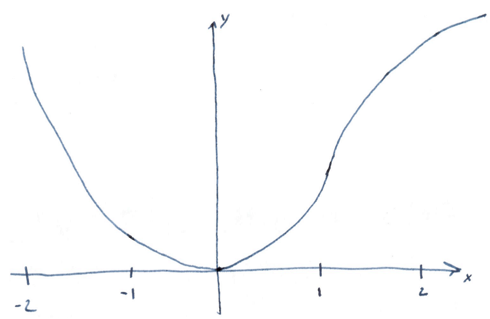
   Answer: Based on the graph, it appears that they are furthest apart after 8 seconds, when they are approximately 30 meters apart.

   (c) At what time do they have the same velocity?

   Answer: The two graphs appear to have the same slope (i.e., velocity) 9 or 10 seconds into the race.

7. Exercise 2.7.20. Sketch the graph of a function g for which $g(0) = g'(0) = 0$, $g'(-1) = -1$, $g'(1) = 3$, $g'(2) = 1$.

   Answer:



8. Exercise 2.7.22. If $g(x) = 1 - x^3$, find $g'(0)$ and use it to find an equation of the tangent line to the curve $y = 1 - x^3$ at the point $(0, 1)$.

   Answer: Since $g(0) = 1$, the slope of the tangent line to the curve $y = 1 - x^3$ is given by $g'(0)$. Now, by definition,

$$g'(0) = \lim_{h \to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0} \frac{\left[1 - (0+h)^3\right] - \left[1 - 0^3\right]}{h} = \lim_{h \to 0} \frac{1 - h^3 - 1}{h} = \lim_{h \to 0} \frac{-h^3}{h}.$$

   Now, so long as $h \neq 0$, we have that $\frac{-h^3}{h} = -h^2$. Therefore,

$$\lim_{h \to 0} \frac{-h^3}{h} = \lim_{h \to 0} \left(-h^2\right) = 0$$

   since $h^2 \to 0$ as $h \to 0$.

   Thus, the tangent line is horizontal; the only horizontal line through $(0, 1)$ is the line $y = 1$, so we see that the equation of the tangent line is $y = 1$.

9. Exercise 2.7.32. The limit

$$\lim_{h\to 0} \frac{\sqrt[4]{16+h}-2}{h}$$

represents the derivative of some function f at some number a. State such an f and a.

Answer: If we let $f(x) = \sqrt[4]{16+x}$ and choose $a = 0$, then

$$\begin{aligned} f'(a) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0} \frac{\sqrt[4]{16+(0+h)}-\sqrt[4]{16+0}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt[4]{16+h}-\sqrt[4]{16}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt[4]{16+h}-2}{h}, \end{aligned}$$

which is the given limit. Therefore, this limit represents $f'(0)$ for $f(x) = \sqrt[4]{16+x}$.

10. Exercise 2.7.48. The quantity (in pounds) of a gourmet ground coffee that is sold by a coffee company at a price of p dollars per pound is $Q = f(p)$.

(a) What is the meaning of the derivative $f'(8)$? What are its units?

Answer: The derivative of a function always gives the change in the output with respect to the change in the input. In this case, since the input to the function is the price and the output is the quantity sold, the derivative gives the change in quantity sold with respect to a change in the price. This is essentially what economists call the "price elasticity of demand".

In particular, $f'(8)$ will give the expected change in quantity sold if the price is changed from \$8/lb. The units are pounds per dollar.

(b) Is $f'(8)$ positive or negative? Explain.

Answer: $f'(8)$ will almost certainly be negative. Remember that

$$f'(8) = \lim_{h\to 0} \frac{f(8+h)-f(8)}{h}.$$

If $h > 0$, then $f(8+h)$ represents the quantity of coffee sold for some price that's greater than \$8/lb., which we would certainly expect to be less than $f(8)$ (the quantity sold when the price is exactly \$8/lb.). Hence, the number $f(8+h) - f(8)$ will be negative and so, since $h > 0$,

$$\frac{f(8+h)-f(8)}{h} < 0.$$

On the other hand, when $h < 0$, the numerator inside the above limit will be the difference between the quantity sold when the price is less than \$8/lb. and the quantity sold when the price is exactly \$8/lb. This will almost certainly be a positive number, so again (since $h < 0$) we'll have

$$\frac{f(8+h)-f(8)}{h} < 0.$$

Hence, the expression inside the limit will always be negative, so we would expect $f'(8)$ to be negative.

Colloquially, this just says that we would expect the quantity sold to go down when the price goes up, and the quantity sold to go up when the price goes down.