

Math 113 HW #4 Solutions

1. Exercise 2.3.14. If it exists, evaluate the limit

$$\lim_{x \rightarrow 4} \frac{x^2 - 4x}{x^2 - 3x - 4}.$$

Answer: Since $x^2 - 4x = x(x - 4)$ and since $x^2 - 3x - 4 = (x + 1)(x - 4)$, we have that

$$\lim_{x \rightarrow 4} \frac{x^2 - 4x}{x^2 - 3x - 4} = \lim_{x \rightarrow 4} \frac{x(x - 4)}{(x + 1)(x - 4)} = \lim_{x \rightarrow 4} \frac{x}{x + 1} = \frac{4}{5}.$$

2. Exercise 2.3.28. Evaluate the limit

$$\lim_{h \rightarrow 0} \frac{(3 + h)^{-1} - 3^{-1}}{h}$$

if it exists.

Answer: Let's focus, for the moment, on the numerator. Finding a common denominator, we see that

$$\frac{1}{3 + h} - \frac{1}{3} = \frac{3}{3(3 + h)} - \frac{3 + h}{3(3 + h)} = \frac{-h}{3(3 + h)}.$$

Therefore, so long as $h \neq 0$, we have that

$$\frac{(3 + h)^{-1} - 3^{-1}}{h} = \frac{\frac{-h}{3(3 + h)}}{h} = \frac{-1}{3(3 + h)} = \frac{-1}{9 + 3h}.$$

Hence,

$$\lim_{h \rightarrow 0} \frac{(3 + h)^{-1} - 3^{-1}}{h} = \lim_{h \rightarrow 0} \frac{-1}{9 + 3h}.$$

But now $\lim_{h \rightarrow 0} (-1) = -1$ and $\lim_{h \rightarrow 0} (9 + 3h) = 9$, so we can use Limit Law #5 to compute

$$\lim_{h \rightarrow 0} \frac{-1}{9 + 3h} = \frac{\lim_{h \rightarrow 0} (-1)}{\lim_{h \rightarrow 0} (9 + 3h)} = -\frac{1}{9}.$$

Putting this all together, we can conclude that

$$\lim_{h \rightarrow 0} \frac{(3 + h)^{-1} - 3^{-1}}{h} = -\frac{1}{9}.$$

3. Show that $\lim_{x \rightarrow 0^+} \sqrt{x}e^x = 0$.

Proof. Using Limit Law #4,

$$\lim_{x \rightarrow 0^+} \sqrt{x}e^x = \left(\lim_{x \rightarrow 0^+} \sqrt{x} \right) \left(\lim_{x \rightarrow 0^+} e^x \right).$$

Since \sqrt{x} approaches 0 as x approaches zero and since e^x approaches 1 as x approaches 0, we see that the right hand side is equal to zero. Therefore,

$$\lim_{x \rightarrow 0^+} \sqrt{x}e^x = 0.$$

□

4. Exercise 2.3.56. If $\lim_{x \rightarrow 0} \frac{f(x)}{x^2} = 5$, find the following limits.

(a) $\lim_{x \rightarrow 0} f(x)$

Answer: The only way I can see how to do this is to re-express what we want in terms of what we know. Since we know $\lim_{x \rightarrow 0} \frac{f(x)}{x^2}$, it would be nice to express $f(x)$ in terms of $\frac{f(x)}{x^2}$. We can do this when $x \neq 0$:

$$f(x) = \frac{f(x)}{x^2} x^2.$$

Therefore,

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \left(\frac{f(x)}{x^2} x^2 \right).$$

But, using one of the limit laws, this is equal to

$$\left(\lim_{x \rightarrow 0} \frac{f(x)}{x^2} \right) \left(\lim_{x \rightarrow 0} x^2 \right) = 5 \cdot 0 = 0.$$

(b) $\lim_{x \rightarrow 0} \frac{f(x)}{x}$

Answer: When $x \neq 0$,

$$\frac{f(x)}{x} = \frac{f(x)}{x^2} x.$$

Therefore,

$$\lim_{x \rightarrow 0} \frac{f(x)}{x} = \lim_{x \rightarrow 0} \left(\frac{f(x)}{x^2} x \right).$$

Using one of the limit laws, this is equal to

$$\left(\lim_{x \rightarrow 0} \frac{f(x)}{x^2} \right) \left(\lim_{x \rightarrow 0} x \right) = 5 \cdot 0 = 0.$$

5. One can prove that, if $f(x) = \frac{e^x - 1}{x}$, then

$$\lim_{x \rightarrow 0} f(x) = 1.$$

How close does x need to be to 0 in order for $f(x)$ to be within 0.5 of the limit 1?

How close does x need to be to 0 in order for $f(x)$ to be within 0.1 of the limit 1?

(In other words, you've found the δ corresponding to the choices $\varepsilon = 0.5$ and $\varepsilon = 0.1$.)

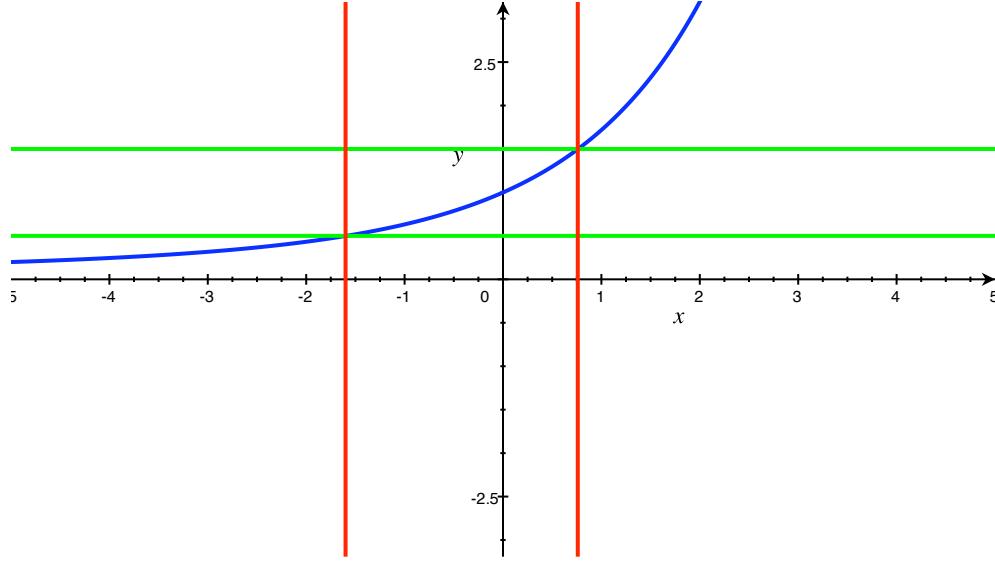
Answer: If we want $f(x)$ to be within 0.5 of the limit 1, that means we want

$$0.5 \leq f(x) \leq 1.5,$$

or, equivalently,

$$0.5 \leq \frac{e^x - 1}{x} \leq 1.5.$$

Shown below are the graph $y = \frac{e^x - 1}{x}$ and the lines $y = 0.5$ and $y = 1.5$ (in green) and $x = -1.6$ and $x = 0.76$ (in red).



For any x between ≈ -1.6 and ≈ 0.76 , we will have that $0.5 \leq f(x) \leq 1.5$. In particular, so long as we choose x to be within a distance of 0.76 of 0 , we will have that $f(x)$ is within a distance of 0.5 of the limit 1 .

If we want that $f(x)$ is within 0.1 of the limit 1 , then that means that we want $0.9 \leq f(x) \leq 1.1$. Similar reasoning to the above demonstrates that this will be the case so long as we choose x within 0.18 of 0 .

6. Exercise 2.4.16. Prove that

$$\lim_{x \rightarrow -2} \left(\frac{1}{2}x + 3 \right) = 2$$

using the ε, δ definition of limit and illustrate with a diagram like Figure 9.

Proof. Suppose $\varepsilon > 0$. Let $\delta = 2\varepsilon$. If

$$0 < |x - (-2)| < \delta,$$

then

$$\begin{aligned} \left| \left(\frac{1}{2}x + 3 \right) - 2 \right| &= \left| \frac{1}{2}x + 1 \right| \\ &= \frac{1}{2}|x + 2| \\ &< \frac{1}{2}\delta \\ &= \frac{1}{2}(2\varepsilon) \\ &= \varepsilon, \end{aligned}$$

where we used the fact that $|x - (-2)| < \delta$ to go from the second to the third lines.

Therefore, we see that $\lim_{x \rightarrow -2} \left(\frac{1}{2}x + 3 \right) = 2$. \square

7. Exercise 2.5.4. From the graph of g , state the intervals on which g is continuous.

Answer: The function g is continuous on the following intervals:

$$[-4, -2), \quad (-2, 2), \quad (2, 4), \quad (4, 6), \quad (6, 8).$$

8. Exercise 2.5.16. Explain why the function

$$f(x) = \begin{cases} \frac{1}{x-1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$$

is discontinuous at $a = 1$. Sketch a graph of the function.

Answer: If f were continuous at 1, then, by the definition of continuity, we would have that

$$\lim_{x \rightarrow 1} f(x) = f(1) = 2.$$

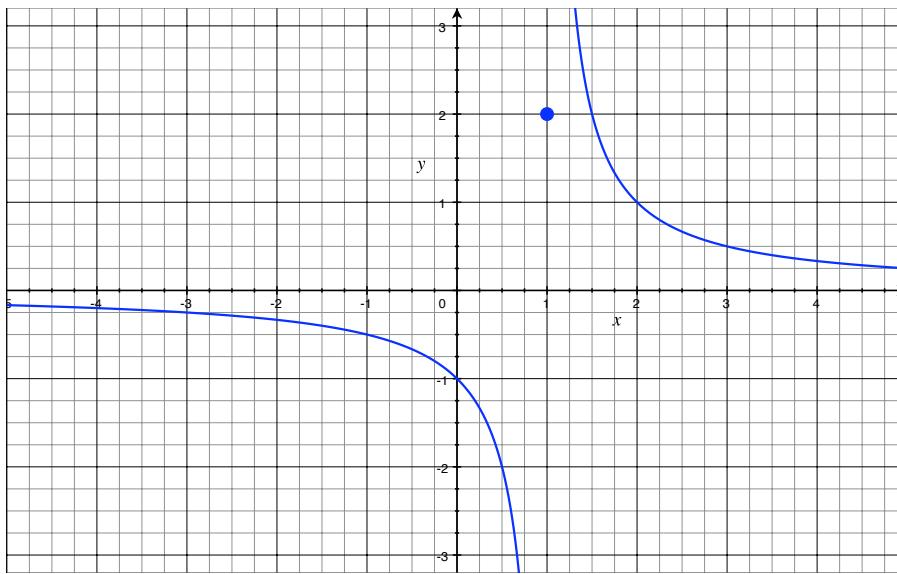
So to show that f is not continuous at 1, we just need to show that the limit on the left hand side is not equal to 2.

In fact, the limit does not exist (and so clearly cannot equal 2) because

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{1}{x-1} = +\infty,$$

whereas

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} \frac{1}{x-1} = -\infty.$$



9. Exercise 2.5.24. Explain, using Theorems 4, 5, 7, and 9, why the function

$$h(x) = \frac{\sin x}{x+1}$$

is continuous at every number in its domain. State the domain.

Answer: The function $\sin x$ is defined for all real numbers x , as is the function $x + 1$. The only problem is when $x = -1$ (when the denominator equals 0), so the domain of h is all real numbers except -1 .

Since $\sin x$ is continuous (Theorem 7) and since $x + 1$ is continuous (Theorem 7), we have that $h(x) = \frac{\sin(x)}{x+1}$ is continuous on its domain by Theorem 4.

10. Exercise 2.5.36. Show that

$$f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geq \pi/4 \end{cases}$$

is continuous on $(-\infty, \infty)$.

Answer: Since $\sin x$ and $\cos x$ are continuous on $(-\infty, \infty)$, the only potential problem will occur at $x = \pi/4$. Now, we need to check that

$$\lim_{x \rightarrow \pi/4} f(x) = f(\pi/4) = \cos(\pi/4) = \frac{1}{\sqrt{2}}$$

(which can also be written as $\frac{\sqrt{2}}{2}$). To see this, we check the limit from each side separately. From the left,

$$\lim_{x \rightarrow \pi/4^-} f(x) = \lim_{x \rightarrow \pi/4^-} \sin x = \sin(\pi/4) = \frac{1}{\sqrt{2}}$$

since $\sin x$ is continuous.

From the right,

$$\lim_{x \rightarrow \pi/4^+} f(x) = \lim_{x \rightarrow \pi/4^+} \cos x = \cos(\pi/4) = \frac{1}{\sqrt{2}}$$

since $\cos x$ is continuous.

Therefore, since the limits from both sides agree and are equal to $\frac{1}{\sqrt{2}}$, we see that

$$\lim_{x \rightarrow \pi/4} f(x) = \frac{1}{\sqrt{2}},$$

so f is indeed continuous.