Math 113 HW #11 Solutions

1. Exercise 4.8.16. Use Newton's method to approximate the positive root of $2 \cos x = x^4$ correct to six decimal places.

   Answer: Let $f(x) = 2 \cos x - x^4$. Then we want to use Newton's method to find the $x > 0$ such that $f(x) = 0$.
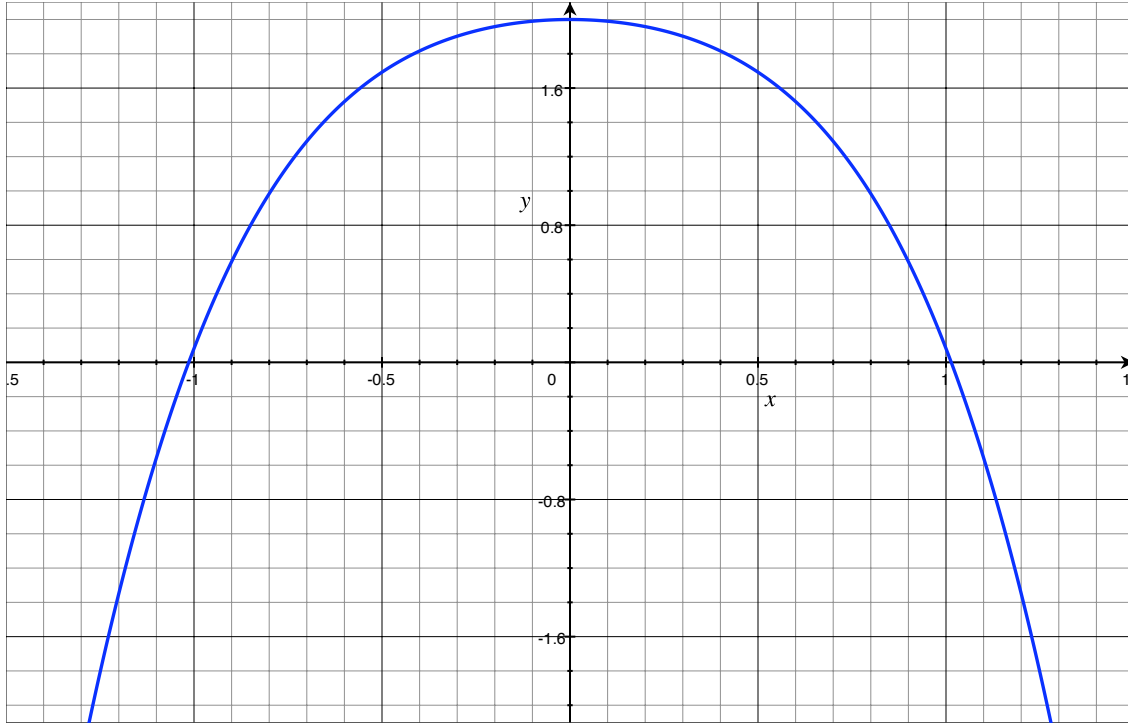
   Notice that
   $$f'(x) = -2 \sin x - 4x^3.$$

Figure 1: $f(x) = 2 \cos x - x^4$

Now, based on the graph of f, guess $x_0 = 1$. Then
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{2 \cos(1) - 1^4}{-2 \sin(1) - 4(1)^3} \approx 1.014183.$$

Then
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 1.014183 - \frac{2 \cos(1.014183) - 1.014183^4}{-2 \sin(1.014183) - 4(1.014183)^3} \approx 1.013957.$$

In turn,
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 1.013957 - \frac{2 \cos(1.013957) - 1.013957^4}{-2 \sin(1.013957) - 4(1.013957)^3} \approx 1.013957,$$

so we can stop, since this is the same as x_2 to six decimal places. Therefore, the positive root of the equation $2 \cos x = x^4$ is, to six decimal places, 1.013957.

2. Exercise 4.8.26. Use Newton's method to find all the roots of the equation $3\sin(x^2) = 2x$ correct to eight decimal places. Start by drawing a graph to find initial approximations.
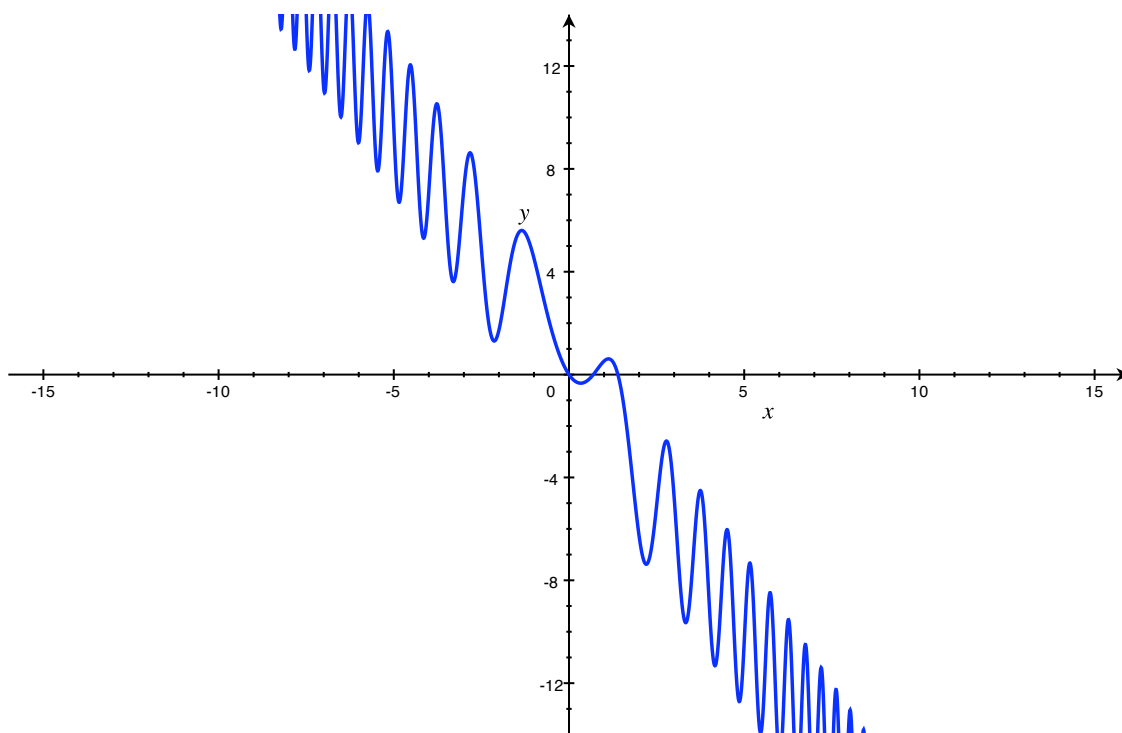
Answer: Let $f(x) = 3\sin(x^2) - 2x$. We want to approximate the values of x such that $f(x) = 0$. We'll need to use the derivative of f, so compute

$$f'(x) = 3\cos(x^2) \cdot 2x - 2 = 6x\cos(x^2) - 2.$$

In general,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3\sin(x_n^2) - 2x_n}{6x_n\cos(x_n^2) - 2}.$$

Now, the graph of f looks like:

From the figure, it appears there are three zeroes of $f(x)$. One seems to be at the origin and, in fact,

$$f(0) = 3\sin(0^2) - 2(0) = 0 - 0 = 0,$$

so one root of the equation is zero.

The next root is approximately $1/2$, so guess $x_0 = 1/2$ and use Newton's Method to compute

the following sequence:

$$x_0 = \frac{1}{2}$$
$$x_1 \approx 0.78430299$$
$$x_2 \approx 0.69609320$$
$$x_3 \approx 0.69300735$$
$$x_4 \approx 0.69299995$$
$$x_5 \approx 0.69299995$$

We can stop here and conclude that, to eight decimal places, the second root of the equation is 0.69299995.

Based on the graph, the last root of f is approximately $3/2$, so start Newton's Method with the guess $x_0 = 3/2$:

$$x_0 = \frac{3}{2}$$
$$x_1 \approx 1.41301039$$
$$x_2 \approx 1.39594392$$
$$x_3 \approx 1.39525190$$
$$x_4 \approx 1.39525077$$
$$x_5 \approx 1.39525077$$

Thus the third root of the equation is, to eight decimal places, 1.39525077.

Putting it all together, we see that, with eight decimal places' accuracy, the three roots of the equation $3\sin(x^2) = 2x$ are

$$0, 0.69299995, 1.39525077.$$

3. Exercise 4.8.30.

   (a) Apply Newton's method to the equation $1/x - a = 0$ to derive the following reciprocal algorithm:
   $$x_{n+1} = 2x_n - ax_n^2.$$
   (This enables a computer to find reciprocals without actually dividing.)

   Answer: Let $f(x) = 1/x - a$. Then the derivative of f is given by

   $$f'(x) = -\frac{1}{x^2},$$

   so the appropriate sequence for Newton's Method is determined by the recurrence rela-

tion

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{\frac{1}{x_n} - a}{-\frac{1}{x_n^2}}$$

$$= x_n - \left(-x_n^2\right)\left(\frac{1}{x_n} - a\right)$$

$$= x_n + x_n - ax_n^2$$

$$= 2x_n - ax_n^2,$$

as desired.

(b) Use part (a) to compute $1/1.6984$ correct to six decimal places.
Answer: Let $a = 1.6984$ in the above expression, so

$$x_{n+1} = 2x_n - 1.6984x_n^2.$$

Now, since 1.6984 is a bit smaller than 2, $1/1.6984$ should be a little bigger than $1/2$, so $x_0 = 1/2$ isn't a bad guess. Then using the above expression to compute $x_1, x_2, \ldots$, we get the sequence

$$x_0 = \frac{1}{2}$$
$$x_1 = 0.5754$$
$$x_2 \approx 0.588484$$
$$x_3 \approx 0.588789$$
$$x_3 \approx 0.588789$$

so, to six decimal places' accuracy, $\frac{1}{1.6984} \approx 0.588789$.

4. Exercise 4.9.10. Find the most general antiderivative of the function

$$f(x) = \sqrt[4]{x^3} + \sqrt[3]{x^4}.$$

Answer: We can re-write f as
$$f(x) = x^{3/4} + x^{4/3}.$$

Then, using the reverse of the power rule, it's easy to see that the following is the most general antiderivative of f:
$$\frac{x^{7/4}}{7/4} + \frac{x^{7/3}}{7/3} + C = \frac{4}{7}\sqrt[4]{x^7} + \frac{3}{7}\sqrt[3]{x^7} + C.$$

5. Exercise 4.9.44. Find f given that

$$f''(t) = 2e^t + 3\sin t, \quad f(0) = 0, \quad f(\pi) = 0.$$

4

Answer: Given what we know about $f''(t)$, we can determine $f'(t)$ as follows:

$$f'(t) = \int f''(t)dt = \int (2e^t + 3\sin t)dt = 2e^t - 3\cos t + C.$$

In turn, this means that

$$f(t) = \int f'(t)dt = \int (2e^t - 3\cos t + C)dt = 2e^t - 3\sin t + Ct + D,$$

where C and D are both (as yet unknown) constants. To determine C and D, we use our knowledge of the values of f. Plugging in $t = 0$, we have that

$$0 = f(0) = 2e^0 - 3\sin(0) + C(0) + D = 2 - 0 + 0 + D = 2 + D,$$

so it must be the case that $D = -2$. Hence,

$$f(t) = 2e^t - 3\sin t + Ct - 2.$$

Now, plugging in $t = \pi$, we have that

$$0 = f(\pi) = 2e^\pi - 3\sin(\pi) + C\pi - 2 = 2e^\pi - 0 + C\pi - 2 = 2e^\pi + C\pi - 2.$$

Therefore, $C = \frac{2-2e^\pi}{\pi}$, so we conclude that

$$f(t) = 2e^t - 3\sin t + \frac{2 - 2e^\pi}{\pi}t - 2.$$

6. **Exercise 4.9.50.** The graph of a function f is shown. Which graph is an antiderivative of f and why?

    Answer: The only graph which can be an antiderivative of f is a. To see this, note that, when f is positive, its antiderivative should be increasing, which eliminates b from consideration. Also, when f is negative, its antiderivative should be decreasing; this eliminates c, which is increasing for all visible x.

7. **Exercise 4.9.60.** A particle is moving with the data

    $$a(t) = \cos t + \sin t, \quad s(0) = 0, \quad v(0) = 5.$$

    Find the position (i.e. $s(t)$) of the particle

    Answer: Since $a(t) = v'(t)$, we can determine v as follows:

    $$v(t) = \int a(t)dt = \int (\cos t + \sin t)dt = \sin t - \cos t + C.$$

    Now, since $v(0) = 5$, we can plug in $t = 0$ to see that

    $$5 = v(0) = \sin(0) - \cos(0) + C = 0 - 1 + C = C - 1,$$

    so $C = 6$ and we have that

    $$v(t) = \sin t - \cos t + 6.$$

5

Now, since $v(t) = s'(t)$, we have that

$$s(t) = \int v(t)dt = \int (\sin t - \cos t + 6)dt = -\cos t - \sin t + 6t + D.$$

Now, plugging in $t = 0$, we have that

$$0 = s(0) = -\cos(0) - \sin(0) + 6(0) + D = -1 - 0 - 0 + D = D - 1,$$

so $D = 1$. Therefore, the position of the particle is given by

$$s(t) = -\cos t - \sin t + 6t + 1.$$