Fall 2009 Math 113 Final Exam Solutions

1. What are the domain and range of the function

$$f(x) = \frac{1+e^x}{1-e^x}?$$

Answer: The function is well-defined everywhere except when the denominator is zero, which happens when

$$0 = 1 - e^x,$$

or, equivalently, $e^x = 0$. This only happens when $x = 0$, so we see that the domain of f is the set of all real numbers $\neq 0$.

As for the range of f, notice that

$$\lim_{x\to 0^-} \frac{1+e^x}{1-e^x} = +\infty$$

and

$$\lim_{x\to 0^+} \frac{1+e^x}{1-e^x} = -\infty.$$

Now,

$$\lim_{x\to -\infty} \frac{1+e^x}{1-e^x} = 1$$

and

$$\lim_{x\to +\infty} \frac{1+e^x}{1-e^x} = \lim_{x\to +\infty} \frac{e^x}{-e^x} = -1$$

by L'Hôpital's Rule, so f has horizontal asymptotes at 1 and -1. Finally, since

$$f'(x) = \frac{(1-e^x)e^x - (1+e^x)(-e^x)}{(1-e^x)^2} = \frac{e^x - e^{2x} + e^x + e^{2x}}{(1-e^x)^2} = \frac{2e^x}{(1-e^x)^2}$$

is never zero, f has no critical points and, thus, no maxima or minima. Putting this all together, the range of f consists of those real numbers x such that

$$-\infty < x < -1 \quad \text{or} \quad 1 < x < +\infty.$$

2. Suppose

$$g(x) = \ln\left(\frac{1}{x} + 1\right).$$

What is $g^{-1}(x)$? For what values of x is $g^{-1}(x)$ defined?

Answer: To determine $g^{-1}(x)$, swap x and y and solve for y:

$$x = \ln\left(\frac{1}{y} + 1\right);$$

exponentiate both sides to get

$$e^x = \frac{1}{y} + 1;$$

subtract 1 from both sides:

$$e^x - 1 = \frac{1}{y};$$

finally, take the reciprocal of both sides:

$$\frac{1}{e^x - 1} = y.$$

Therefore,

$$g^{-1}(x) = \frac{1}{e^x - 1}.$$

Clearly, $g^{-1}(x)$ is only defined for $x \neq 0$.

3. What is the equation of the line tangent to the graph of $y^3 + 3x^2y^2 + 2x^3 = 4$ at the point $(1, -1)$?

   Answer: The goal is to determine y' by using implicit differentiation.

   Differentiating both sides yields:

$$3y^2 \cdot y' + 6xy^2 + 3x^2 \cdot 2y \cdot y' + 6x^2 = 0,$$

   or, equivalently,

$$y'(3y^2 + 6x^2y) + 6xy^2 + 6x^2 = 0.$$

   Hence,

$$y' = \frac{-6xy^2 - 6x^2)}{3y^2 + 6x^2y}.$$

   Therefore, the slope of the tangent line at $(1, -1)$ will be

$$\frac{-6(1)(-1)^2 - 6(1)^2}{3(-1)^2 + 6(1)^2(-1)} = \frac{-12}{-3} = 4.$$

   Hence, using the point-slope formula, the tangent line at $(1, -1)$ will be

$$y + 1 = 4(x - 1) = 4x - 4,$$

   or, equivalently,

$$y = 4x - 5.$$

4. Evaluate the limit

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2 + x}.$$

   Answer: Notice that both numerator and denominator go to zero as $x \to 0$. Hence, we can apply L'Hopital's Rule:

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2 + x} = \lim_{x \to 0} \frac{\sin x}{2x + 1} = 0,$$

   since $\sin(0) = 0$.

5. The volume of a cube is increasing at a rate of 10 cm^3/min. How fast is the surface area increasing when the length of an edge is 10 cm?

   Answer: If $V(t)$ is the volume of the cube after t minutes, $A(t)$ is the surface area, and $\ell(t)$ is the length of an edge, then we know that

$$\begin{aligned} V(t) &= (s(t))^3 \\ A(t) &= 6(s(t))^2 \\ V'(t) &= 10 \text{ for all } t \\ s(t_0) &= 10 \text{ where } t_0 is the moment of interest \end{aligned}$$

   and we're asked to figure out $A'(t_0)$. Now, we know that

$$A'(t) = 6 \cdot 2s(t) \cdot s'(t) = 12s(t)s'(t),$$

so

$$A'(t_0) = 12s(t_0)s'(t_0) = 12(10)s'(t_0) = 120s'(t_0),$$

so all we need to do is determine $s'(t_0)$. To do so, let's differentiate V:

$$V'(t) = 3(s(t))^2 \cdot s'(t).$$

Then, plugging in $t = t_0$, we have

$$10 = V'(t_0) = 3(s(t_0))^2 \cdot s'(t_0) = 3(10)^2 \cdot s'(t_0) = 300s'(t_0),$$

meaning that $s'(t_0) = \frac{10}{300} = \frac{1}{30}$. Therefore, we know that

$$A'(t_0) = 120s'(t_0) = 120 \cdot \frac{1}{30} = 4,$$

so the surface area is increasing at a rate of 4 cm^2/sec.

6. Find the maximum and minimum values, inflection points and asymptotes of $y = \ln(x^2 + 1)$ and use this information to sketch the graph.

Answer: Notice that

$$y' = \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1}$$

and, by the Quotient Rule,

$$y'' = \frac{(x^2+1)(2) - 2x(2x)}{(x^2+1)^2} = \frac{2x^2 + 2 - 4x^2}{(x^2+1)^2} = \frac{2-2x^2}{(x^2+1)^2}.$$

Now, the critical points occur when $y' = 0$, which is to say when

$$\frac{2x}{x^2+1} = 0.$$

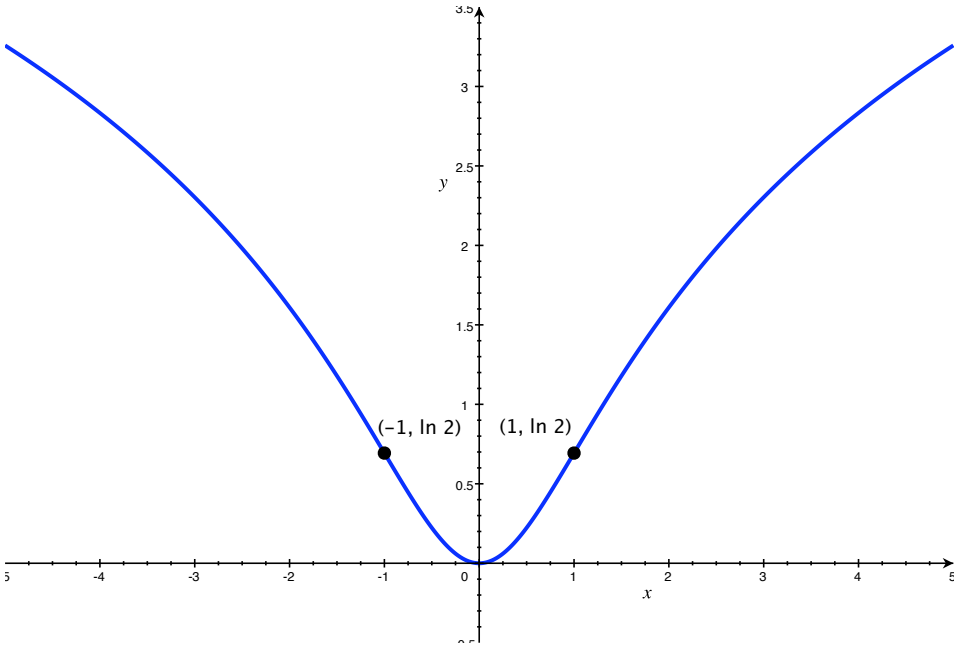
The only happens when $x = 0$, so 0 is the only critical point. Notice that $y''(0) = 2$, which is greater than zero, so the second derivative test implies that 0 is a local minimum.

$y'' = 0$ when $2 - 2x^2 = 0$, meaning when $x = \pm 1$, so there are inflection points at $x = \pm 1$. Finally,

$$\lim_{x \to -\infty} \ln(x^2+1) = \infty = \lim_{x \to +\infty} \ln(x^2+1),$$

so there are no horizontal asymptotes.

Putting all this together, we see that y has a minimum at 0 and is concave up between -1 and 1 and concave down everywhere else and has no asymptotes, meaning that the graph looks something like this:



7. Use an appropriate linearization to approximate $e^{1/10}$.

   Answer: Consider the function $f(x) = e^x$. Then I want to approximate $f(1/10) = e^{1/10}$ using the linearization of f at $x = 0$. This linearization is given by

$$L(x) = f(0) + f'(0)(x - 0) = f(0) + f'(0)x.$$

   Now $f(0) = e^0 = 1$ and $f'(x) = e^x$, so $f'(0) = e^0 = 1$. Hence, the linearization is given by

$$L(x) = 1 + x.$$

   Therefore,

$$f(1/10) \approx L(1/10) = 1 + \frac{1}{10} = \frac{11}{10} = 1.1.$$

8. What is the absolute maximum value of $f(x) = x^{1/x}$ for $x > 0$?

   Answer: Taking the natural log of both sides,

$$\ln f(x) = \ln(x^{1/x}) = \frac{1}{x} \ln x$$

   . Now differentiating, we see that

$$\frac{f'(x)}{f(x)} = \frac{1}{x} \cdot \frac{1}{x} - \frac{1}{x^2} \ln x = \frac{1}{x^2}(1 - \ln x),$$

   so

$$f'(x) = f(x)\frac{1}{x^2}(1 - \ln x) = \frac{x^{1/x}}{x^2}(1 - \ln x).$$

   Since $x^{1/x}$ is never zero for $x > 0$, $f'(x) = 0$ only when $1 - \ln x = 0$, meaning that $\ln x = 1$. This only happens when $x = e$, so e is the only critical point of f. Notice that $f'(x)$ changes sign from positive to negative at $x = e$, so the first derivative test implies that f has a local maximum at e. However, since this is the only critical point and there are no endpoints, this must, in fact, be the global maximum of f.

Hence, the absolute maximum value of $f(x)$ for $x > 0$ is

$$f(e) = e^{1/e}.$$

9. A stock market analyst sold a monthly newsletter to 320 subscribers at a price of \$10 each. She discovered that for each \$0.25 increase in the monthly price of the newsletter, she would lose 2 subscriptions. If she sets the price of the newsletters to bring in the greatest total monthly revenue, what will that revenue be?

   Answer: Assuming the demand function is linear, we know it is given by the line of slope

   $$\frac{0.25}{-2} = -\frac{1}{8}$$

   passing through the point $(320, 10)$. This is the line

   $$y - 10 = -\frac{1}{8}(x - 320) = -\frac{x}{8} + 40$$

   or, equivalently, $y = -\frac{x}{8} + 50$. Hence, the demand function is

   $$p(x) = -\frac{x}{8} + 50.$$

   Therefore, the revenue function is

   $$R(x) = xp(x) = x\left(-\frac{x}{8} + 50\right) = -\frac{x^2}{8} + 50x.$$

   The critical points of the revenue function occur when $R'(x) = 0$. Hence, since

   $$R'(x) = -\frac{x}{4} + 50,$$

   the critical points occur when $-\frac{x}{4} + 50 = 0$, meaning that $x = 200$. Since $R''(x) = -1/4$, we know that this is a maximum.

   Hence, the revenue is maximized when the analyst sells 200 subscriptions. Hence, she should charge

   $$p(200) = -\frac{200}{8} + 50 = 25$$

   dollars per subscription. This will bring in a maximum revenue of

   $$R(200) = 200 \cdot p(200) = 200 \cdot 25 = 5000$$

   dollars per month.

10. Does $\log_3 x$ grow faster than, slower than, or at the same rate as $\log_{10} x$?

    Answer: Taking

    $$\lim_{x\to\infty} \frac{\log_3 x}{\log_{10} x},$$

    notice that, by the properties of logarithms,

    $$\log_3 x = \frac{\ln x}{\ln 3} \quad \text{and} \quad \log_{10} x = \frac{\ln x}{\ln 10}.$$

    Hence, the above limit is equal to

    $$\lim_{x\to\infty} \frac{\frac{\ln x}{\ln 3}}{\frac{\ln x}{\ln 10}} = \lim_{x\to\infty} \frac{\ln 10}{\ln 3} = \frac{\ln 10}{\ln 3}$$

    In turn, $\frac{\ln 10}{\ln 3} = \log_3 10$, which is a little bigger than 2. Hence $\log_3 x$ grows about twice as fast as $\log_{10} x$.

11. Suppose the velocity of a particle is given by

$$v(t) = 3\cos t + 4\sin t.$$

If the particle starts (at time 0) at a position 7 units to the right of the origin, what is the position of the particle at time t?

Answer: Let $s(t)$ be the position of the particle at time t. Then we know that $s'(t) = v(t)$ and that $s(0) = 7$. Now,

$$\int v(t)dt = \int (3\cos t + 4\sin t)dt = 3\sin t - 4\cos t + C.$$

Therefore, since $s(t)$ is *an* antiderivative of $v(t) = s'(t)$, we know that

$$s(t) = 3\sin t - 4\cos t + C$$

for some real number C. To solve for C, plug in $t = 0$:

$$7 = s(0) = 3\sin(0) - 4\cos(0) + C = -4 + C,$$

so we see that $C = 11$.

Therefore, the position of the particle is given by

$$s(t) = 3\sin t - 4\cos t + 11.$$

12. Evaluate the definite integral

$$\int_0^{\pi/6} \frac{2 + \cos^3\theta}{\cos^2\theta} d\theta.$$

Answer: Note that

$$\frac{2 + \cos^3\theta}{\cos^2\theta} = \frac{2}{\cos^2\theta} + \cos\theta = 2\sec^2\theta + \cos\theta.$$

Therefore,

$$\int_0^{\pi/6} \frac{2 + \cos^3\theta}{\cos^2\theta} d\theta = \int_0^{\pi/6} (2\sec^2\theta + \cos\theta)d\theta.$$

By the Fundamental Theorem of Calculus, this is equal to

$$[2\tan\theta + \sin\theta]_0^{\pi/6} = (2\tan(\pi/6) + \sin(\pi/6)) - (2\tan(0) + \sin(0)) = \frac{2}{\sqrt{3}} + \frac{1}{2}.$$

13. Evaluate

$$\int \csc r \cot r \, dr.$$

Answer: Recall that the derivative of $\csc r$ is $-\csc r \cot r$, so

$$\int \csc r \cot r \, dr = -\csc r + C.$$

14. Let

$$g(x) = \int_1^{x^2} \frac{\sin t}{\sqrt{t}} dt.$$

What is the derivative of g?

Answer: Let $h(u) = \int_1^u \frac{\sin t}{\sqrt{t}} dt$ and let $f(x) = x^2$. Then

$$g(x) = h(f(x)),$$

so we can compute $g'(x)$ using the Chain Rule:

$$g'(x) = h'(f(x))f'(x).$$

Now,

$$f'(x) = 2x$$

and, by the Fundamental Theorem of Calculus,

$$h'(u) = \frac{\sin u}{\sqrt{u}}.$$

Hence,

$$g'(x) = h'(f(x))f'(x) = h'(x^2) \cdot 2x = \frac{\sin(x^2)}{\sqrt{x^2}} \cdot 2x = \frac{2x \sin(x^2)}{x} = 2\sin(x^2).$$