Math 113 Exam #1 Solutions

1. What are the domain and range of the function $f(x) = \frac{1}{\sqrt{4-x}}$?

   Answer: $f(x)$ is well-defined provided $\sqrt{4-x)} \neq 0$. In order for $\sqrt{4-x}$ to exist, we must have

$$4 - x \geq 0,$$

   meaning that $x \leq 4$. In order for $\sqrt{4-x} \neq 0$ it must also be the case that $x \neq 4$, so the domain of f is

$$(-\infty, 4).$$

   Since $\sqrt{4-x} > 0$ where f is defined, we see that $f(x) > 0$ for all x in the domain, so the range of f is

$$(0, +\infty).$$

2. Evaluate

$$\lim_{x\to\infty} \frac{\sqrt{4x^2 - 8x + 7}}{17x + 12}$$

   Answer: Dividing both numerator and denominator by x yields

$$\begin{aligned}
\lim_{x\to\infty} \frac{\frac{1}{x}\sqrt{4x^2 - 8x + 7}}{\frac{1}{x}(17x + 12)} &= \lim_{x\to\infty} \frac{\sqrt{\frac{1}{x^2}\left(4x^2 - 8x + 7\right)}}{17 + \frac{12}{x}} \\
&= \lim_{x\to\infty} \frac{\sqrt{4 - \frac{8}{x} + \frac{7}{x^2}}}{17 + \frac{12}{x}} \\
&= \frac{\sqrt{4}}{17} \\
&= \frac{2}{17}.
\end{aligned}$$

3. Let

$$f(x) = \begin{cases} \frac{x-2}{x^2-4} & \text{for } x \neq 2 \\ a & \text{for } x = 2 \end{cases}$$

   If $f(x)$ is continuous at $x = 2$, then find the value of a.

   Answer: In order for f to be continuous at $x = 2$, we must have that

$$a = f(2) = \lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x-2}{x^2-4}.$$

   Now, we can factor the denominator in the limit to get

$$\lim_{x\to 2} \frac{x-2}{x^2-4} = \lim_{x\to 2} \frac{x-2}{(x+2)(x-2)} = \lim_{x\to 2} \frac{1}{x+2} = \frac{1}{4},$$

   so we see that, in order for f to be continuous, a must be $\frac{1}{4}$.

4. Are there any solutions to the equation $\cos x = x$?

Answer: Yes, there is a solution to the equation. To see this, let

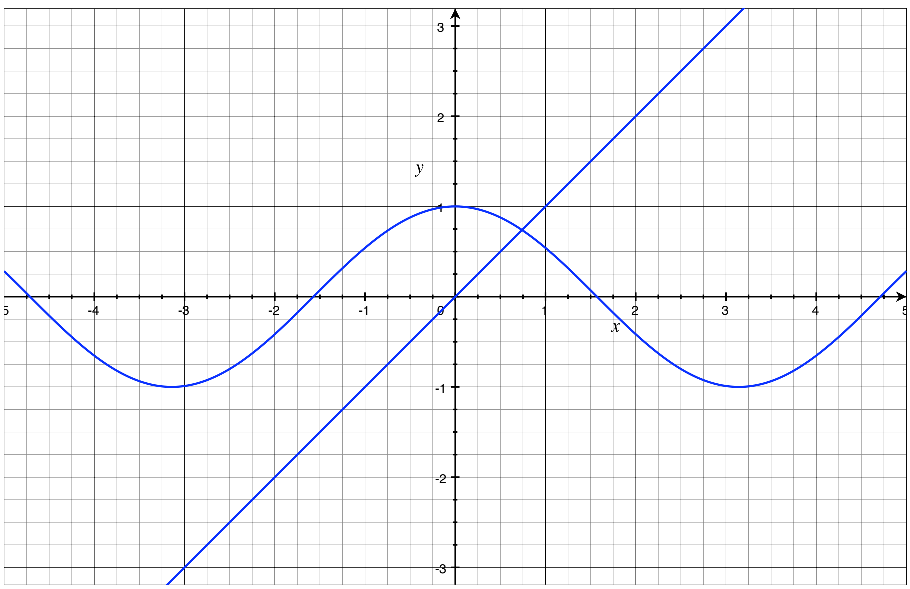
$$f(x) = \cos x - x.$$

Then $f(x) = 0$ precisely when x is a solution to the equation $\cos x = x$, so the problem is to show that $f(x) = 0$ for some x. To see this, notice that f is continuous (since both $\cos x$ and x are continuous functions) and

$$f(0) = \cos 0 - 0 = 1 > 0$$
$$f(\pi) = \cos \pi - \pi = -1 - \pi < 0.$$

Therefore, by the Intermediate Value Theorem there exists c between 0 and π such that $f(c) = 0$. Then, as noted above, $\cos c = c$, so c is a solution to the equation.

Intuitively, we can see that $\cos x = x$ has a solution by looking at the graphs of $y = \cos x$ and $y = x$; they intersect in exactly one point, so the solution c that we saw exists actually be the only solution.



5. Determine the following limits, if they exist

(a) $\lim_{x\to -1} \frac{x^2-2x+1}{x-1}$

Answer: Plugging in $x = -1$ yields that

$$\lim_{x\to -1} \frac{x^2 - 2x + 1}{x - 1} = \frac{(-1)^2 - 2(-1) + 1}{(-1) - 1} = \frac{4}{-2} = -2.$$

(b) $\lim_{x\to 1^-} \frac{x^2+2x+1}{x-1}$

Answer: Notice that

$$\lim_{x\to 1} \left(x^2 + 2x + 1\right) = 1 + 2 + 1 = 4,$$

so the numerator is going to 4. Also, the denominator is going to zero, so we expect the limit to be $\pm\infty$. To see which, notice that, if $x < 1$, then

$$x - 1 < 0,$$

so the denominator is a very small negative number as $x \to 1^-$. Hence,

$$\lim_{x \to 1^-} \frac{x^2 + 2x + 1}{x - 1} = -\infty.$$

6. Let

$$g(x) = \sqrt{x}$$

Is g differentiable at 0? If so, what is $g'(0)$?

Answer: If g is differentiable at 0, then, by definition,

$$\begin{aligned} g'(0) = \lim_{h \to 0} \frac{g(0+h) - g(0)}{h} &= \lim_{h \to 0} \frac{\sqrt{0+h} - \sqrt{0}}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{h}}{h} \\ &= \lim_{h \to 0} \frac{1}{\sqrt{h}}, \end{aligned}$$

which does not exist (when $h > 0$, this goes to $+\infty$; when $h < 0$, the expression $\sqrt{h}$ doesn't even make sense in the real numbers). Therefore, g is *not* differentiable at 0.

7. Let

$$f(x) = 2x^2 + 3x.$$

Is f differentiable at 1? If so, what is $f'(1)$?

Answer: If f is differentiable at 1, then, by definition,

$$\begin{aligned} f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} &= \lim_{h \to 0} \frac{\left[2(1+h)^2 + 3(1+h)\right] - \left[2(1)^2 + 3(1)\right]}{h} \\ &= \lim_{h \to 0} \frac{\left[2(1+2h+h^2) + 3(1) + 3h\right] - \left[2(1)^2 + 3(1)\right]}{h} \\ &= \lim_{h \to 0} \frac{7h + h^2}{h} \\ &= \lim_{h \to 0} (7 + h) \\ &= 7, \end{aligned}$$

so we see that f is differentiable at 1 and that $f'(1) = 7$.