Math 113 HW #5 Solutions

§3.1

22. Differentiate the function

$$y = \sqrt{x}(x-1)$$

Answer: Re-write the function as

$$y = x\sqrt{x} - \sqrt{x} = x^{3/2} - x^{1/2}.$$

Then, using the power rule,

$$y' = \frac{3}{2}x^{1/2} - \frac{1}{2}x^{-1/2} = \frac{3}{2}\sqrt{x} - \frac{1}{2\sqrt{x}}.$$

34. Find an equation of the tangent line to the curve

$$y = x^4 + 2x^2 - x$$

at the point $(1, 2)$.

Answer: Using the power rule,

$$y' = 4x^3 + 4x - 1,$$

so, at the point $(1, 2)$,

$$y' = 4(1)^3 + 4(1) - 1 = 7.$$

Therefore the slope of the tangent line is 7. Since the tangent line passes through the point $(1, 2)$, we can use the point-slope formula:

$$y - 2 = 7(x - 1),$$

so, adding 2 to both sides, we have that

$$y = 7x + 5.$$

64. The equation $y'' + y' - 2y = x^2$ is called a **differential equation** because it involves an unknown function y and its derivatives y' and y''. Find constants A, B and C such that the function $y = Ax^2 + Bx + C$ satisfies this equation.

Answer: Given the general form for y, we know that its derivatives must be:

$$y' = 2Ax + B$$
$$y'' = 2A.$$

Plugging these into the given equation, we have that

$$x^2 = y'' + y' - 2y = (2A) + (2Ax + B) - 2(Ax^2 + Bx + C).$$

Re-arranging the terms on the right hand side, this implies that

$$x^2 = -2Ax^2 + (2A - 2B)x + (2A + B - 2C).$$

Therefore, since the coefficients on x^2 must be equal, we have that $1 = -2A$, meaning that $A = -\frac{1}{2}$. Also, since the coefficients on x must be equal, we have that

$$0 = 2A - 2B = 2\left(-\frac{1}{2}\right) - 2B = -1 - 2B,$$

so $2B = -1$, meaning that $B = -\frac{1}{2}$ as well. Finally, since the constant terms on both sides of the equation must be equal, we have that

$$0 = 2A + B - 2C = 2\left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right) - 2C = -\frac{3}{2} - 2C.$$

Therefore, $2C = -\frac{3}{2}$, meaning that $C = -\frac{3}{4}$.

Hence, we can conclude that

$$y = -\frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}.$$

§3.2

24. Differentiate

$$f(x) = \frac{1 - xe^x}{x + e^x}.$$

Answer: Using the quotient and product rules,

$$\begin{aligned} f'(x) &= \frac{(x + e^x)(-e^x - xe^x) - (1 - xe^x)(1 + e^x)}{(x + e^x)^2} \\ &= \frac{-xe^x - x^2e^{2x} - e^{2x} - xe^{2x} - (1 + e^x - xe^x - xe^{2x})}{(x + e^x)^2} \\ &= \frac{-1 - e^x - e^{2x} - x^2e^{2x}}{(x + e^x)^2} \end{aligned}$$

28. Let

$$f(x) = x^{5/2}e^x$$

and find $f'(x)$ and $f''(x)$.

Answer: Using the product rule,

$$f'(x) = \left(\frac{5}{2}x^{3/2}\right)e^x + x^{5/2}e^x = \frac{5}{2}x^{3/2}e^x + x^{5/2}e^x. \tag{1}$$

Taking the derivative of the first term yields

$$\begin{aligned} \frac{5}{2}\left(\frac{3}{2}x^{1/2}e^x + x^{3/2}e^x + \frac{5}{2}x^{3/2}e^x + x^{5/2}e^x\right) &= \frac{15}{4}x^{1/2}e^x + \left(\frac{5}{2} + \frac{25}{4}x^{3/2}e^x\right) + \frac{5}{2}x^{5/2}e^x \\ &= \frac{15}{4}x^{1/2}e^x + \frac{35}{4}x^{3/2}e^x + \frac{5}{2}x^{5/2}e^x. \end{aligned}$$

On the other hand, the second term in (1) is just a copy of f, so its derivative is the same as $f'(x)$. Therefore,

$$\begin{aligned} f''(x) &= \left(\frac{15}{4}x^{1/2}e^x + \frac{45}{4}x^{3/2}e^x + \frac{5}{2}x^{5/2}e^x\right) + f'(x) \\ &= \left(\frac{15}{4}x^{1/2}e^x + \frac{45}{4}x^{3/2}e^x + \frac{5}{2}x^{5/2}e^x\right) + \left(\frac{5}{2}x^{3/2}e^x + x^{5/2}e^x\right) \\ &= \frac{15}{4}x^{1/2}e^x + \frac{45}{4}x^{3/2}e^x + \frac{7}{2}x^{5/2}e^x. \end{aligned}$$

34. Find the equations of the tangent line and normal line to the curve

$$y = \frac{\sqrt{x}}{x+1}$$

at the point $(4, 0.4)$.

Answer: Writing $\sqrt{x}$ as $x^{1/2}$ and using the quotient rule,

$$\begin{aligned} y' &= \frac{(x+1)\left(\frac{1}{2}x^{-1/2}\right) - x^{1/2}(1)}{(x+1)^2} \\ &= \frac{\frac{1}{2}x^{1/2} + \frac{1}{2}x^{-1/2} - x^{1/2}}{(x+1)^2} \\ &= \frac{1}{2}\frac{x^{-1/2} - x^{1/2}}{(x+1)^2} \\ &= \frac{1}{2}\frac{\frac{1}{\sqrt{x}} - \sqrt{x}}{(x+1)^2} \end{aligned}$$

Therefore, we can plug in $x = 4$ to get the slope of the tangent line at $(4, 0.4)$:

$$\frac{1}{2}\frac{\frac{1}{\sqrt{4}} - \sqrt{4}}{(4+1)^2} = \frac{1}{2}\frac{\frac{1}{2} - 2}{25} = -\frac{3}{100}.$$

Therefore, we can use the point-slope formula to get the equation of the tangent line:

$$y - \frac{2}{5} = -\frac{3}{100}(x-4),$$

so the tangent line is

$$y = -\frac{3}{100}x + \frac{13}{25}.$$

On the other hand, the slope of the normal line is the negative reciprocal of the slope of the tangent line, so the slope of the normal line is $\frac{100}{3}$. Therefore, the point-slope formula tells us that the normal line is

$$y - \frac{2}{5} = \frac{100}{3}(x-4),$$

or

$$y = \frac{100}{3}x - \frac{1994}{15}$$

§3.3

10. Differentiate

$$y = \frac{1+\sin x}{x+\cos x}$$

Answer: Using the quotient rule and what we know about the derivatives of sines and cosines,

$$\begin{aligned} y' &= \frac{(x+\cos x)\cos x - (1+\sin x)(1-\sin x)}{(x+\cos x)^2} \\ &= \frac{x\cos x + \cos^2 x - (1+\sin x - \sin x - \sin^2 x)}{(x+\cos x)^2} \\ &= \frac{x\cos x + \cos^2 x - 1 + \sin^2 x}{(x+\cos x)^2}. \end{aligned}$$

Using the fact that $\sin^2 x + \cos^2 x = 1$, this simplifies as

$$y' = \frac{x\cos x}{(x+\cos x)^2}.$$

22. Find an equation of the tangent line to the curve

$$y = e^x \cos x$$

at the point $(0, 1)$.

Answer: Using the product rule,

$$y' = e^x \cos x + e^x(-\sin x) = e^x \cos x - e^x \sin x.$$

Therefore, plugging in $x = 0$ gives the slope of the tangent line at $(0, 1)$:

$$e^0 \cos 0 - e^0 \sin 0 = 1 - 0 = 1.$$

Therefore, by the point-slope formula, the equation of the tangent line is

$$y - 1 = 1(x - 0),$$

or

$$y = x + 1.$$

34. Find the points on the curve $y = (\cos x)/(2 + \sin x)$ at which the tangent is horizontal.

Answer: The tangent being horizontal means that the slope of the tangent line is zero. In other words, we're looking for those points where the derivative of the function is zero. Using the quotient rule, the derivative is

$$y' = \frac{(2+\sin x)(-\sin x) - \cos x(\cos x)}{(2+\sin x)^2} = \frac{-2\sin x - \sin^2 x - \cos^2 x}{(2+\sin x)^2}.$$

Using the fact that $\sin^2 x + \cos^2 x = 1$, this simplifies to

$$y' = \frac{-2\sin x - 1}{(2+\sin x)^2},$$

which is zero if and only if the numerator is zero. Therefore, we want to find the values of x for which $0 = -2\sin x - 1$ or, equivalently,

$$\sin x = -\frac{1}{2}.$$

This equality holds when

$$x = \ldots, \frac{-2\pi}{3}, \frac{-\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3}, \frac{11\pi}{3}, \ldots$$

§3.4

12. Find the derivative of

$$f(t) = \sqrt[3]{1+\tan t}.$$

Answer: Re-writing as

$$f(t) = (1+\tan t)^{1/3},$$

the Chain Rule tells us that

$$f'(t) = \frac{1}{3}(1+\tan t)^{-2/3}(\sec^2 t) = \frac{\sec^2 t}{3(1+\tan t)^{2/3}}.$$

22. Find the derivative of the function

$$y = e^{-5x}\cos 3x.$$

Answer: By the Chain Rule, the derivative of e^{-5x} is

$$e^{-5x}(-5) = -5e^{-5x}.$$

Also using the Chain Rule, the derivative of $\cos 3x$ is

$$-\sin(3x)(3) = -3\sin 3x.$$

Therefore, using these facts and the product rule,

$$y' = -5e^{-5x}\cos 3x + e^{-5x}(-3\sin 3x) = -5e^{-5x}\cos 3x - 3e^{-5x}\sin 3x.$$

34. Find the derivative of the function

$$y = x\sin\frac{1}{x}.$$

Answer: By the Chain Rule, the derive of $\sin\frac{1}{x}$ is

$$\cos\frac{1}{x}\left(-x^{-2}\right) = -\frac{\cos\frac{1}{x}}{x^2}.$$

Therefore, using the product rule,

$$y' = \sin\frac{1}{x} - \frac{\cos\frac{1}{x}}{x^2}.$$

40. Find the derivative of the function

$$y = \sin(\sin(\sin x)).$$

Answer: Using the Chain Rule,

$$\begin{aligned} y' &= \cos(\sin(\sin x)) \frac{d}{dx}(\sin(\sin x)) \\ &= \cos(\sin(\sin x)) \cos(\sin x) \frac{d}{dx}(\sin x) \\ &= \cos(\sin(\sin x)) \cos(\sin x) \cos x. \end{aligned}$$