Math 113 HW #4 Solutions

§2.4

16. Prove that

$$\lim_{x \to -2} \left(\frac{1}{2}x + 3 \right) = 2$$

using the ε, δ definition of limit and illustrate with a diagram like Figure 9.

Proof. Suppose $\varepsilon > 0$. Let $\delta = 2\varepsilon$. If

$$0 < |x - (-2)| < \delta,$$

then

$$\begin{aligned} \left| \left(\frac{1}{2}x + 3 \right) - 2 \right| &= \left| \frac{1}{2}x + 1 \right| \\ &= \frac{1}{2} |x + 2| \\ &< \frac{1}{2}\delta \\ &= \frac{1}{2}(2\varepsilon) \\ &= \varepsilon, \end{aligned}$$

where we used the fact that $|x - (-2)| < \delta$ to go from the second to the third lines. Therefore, we see that $\lim_{x \to -2} \left(\frac{1}{2}x + 3 \right) = 2$. □

24. Prove that

$$\lim_{x \to a} c = c$$

using the ε, δ definition of limit.

Proof. Suppose $\varepsilon > 0$. It's going to turn out that any possible choice of δ is going to work, I'll pick $\delta = 1$. If

$$0 < |x - a| < \delta,$$

then

$$|c - c| = 0 < \epsilon,$$

so $\lim_{x \to a} c = c$. □

§2.5

4. From the graph of g, state the intervals on which g is continuous.

Answer: The function g is continuous on the following intervals:

$$[-4,-2), \quad (-2,2), \quad (2,4), \quad (4,6), \quad (6,8).$$

16. Explain why the function

$$f(x) = \begin{cases} \frac{1}{x-1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$$

is discontinuous at $a = 1$. Sketch a graph of the function.

Answer: If f were continuous at 1, then, by the definition of continuity, we would have that

$$\lim_{x\to 1} f(x) = f(1) = 2.$$

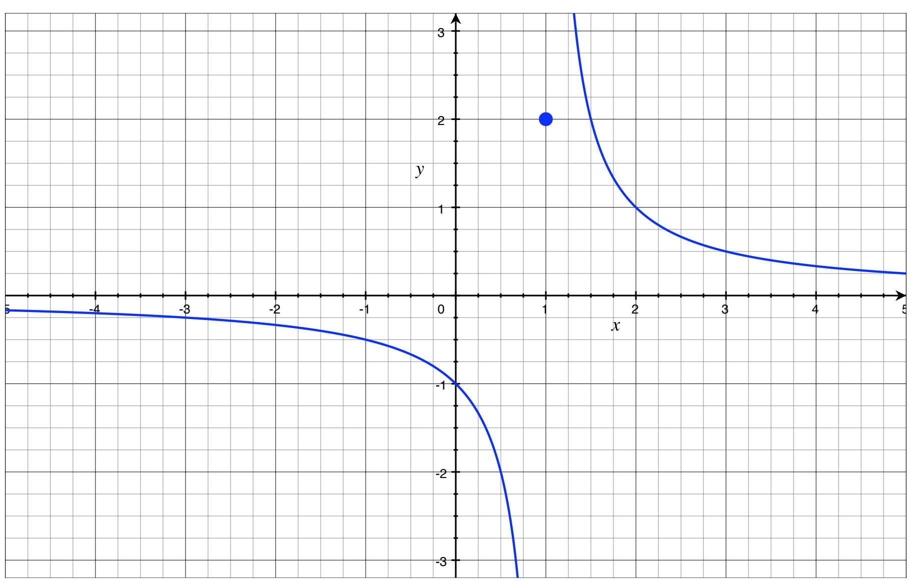
So to show that f is not continuous at 1, we just need to show that the limit on the left hand side is not equal to 2.

In fact, the limit does not exist (and so clearly cannot equal 2) because

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{1}{x-1} = +\infty,$$

whereas

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{1}{x-1} = -\infty.$$



24. Explain, using Theorems 4, 5, 7, and 9, why the function

$$h(x) = \frac{\sin x}{x+1}$$

is continuous at every number in its domain. State the domain.

Answer: The function $\sin x$ is defined for all real numbers x, as is the function $x+1$. The only problem is when $x = -1$ (when the denominator equals 0), so the domain of h is all real numbers except -1.

Since $\sin x$ is continuous (Theorem 7) and since $x+1$ is continuous (Theorem 7), we have that $h(x) = \frac{\sin(x)}{x+1}$ is continuous on its domain by Theorem 4.

36. Show that

$$f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x \text{ if } x \geq \pi/4 \end{cases}$$

is continuous on $(-\infty, \infty)$.

Answer: Since $\sin x$ and $\cos x$ are continuous on $(-\infty, \infty)$, the only potential problem will occur at $x = \pi/4$. Now, we need to check that

$$\lim_{x \to \pi/4} f(x) = f(\pi/4) = \cos(\pi/4) = \frac{1}{\sqrt{2}}$$

(which can also be written as $\frac{\sqrt{2}}{2}$). To see this, we check the limit from each side separately. From the left,

$$\lim_{x \to \pi/4^-} f(x) = \lim_{x \to \pi/4^-} \sin x = \sin(\pi/4) = \frac{1}{\sqrt{2}}$$

since $\sin x$ is continuous.

From the right,

$$\lim_{x \to \pi/4^+} f(x) = \lim_{x \to \pi/4^+} \cos x = \cos(\pi/4) = \frac{1}{\sqrt{2}}$$

since $\cos x$ is continuous.

Therefore, since the limits from both sides agree and are equal to $\frac{1}{\sqrt{2}}$, we see that

$$\lim_{x \to \pi/4} f(x) = \frac{1}{\sqrt{2}},$$

so f is indeed continuous.

48. Use the Intermediate Value Theorem to show that there is a solution of the equation

$$\sqrt[3]{x} = 1 - x$$

in the interval $(0, 1)$.

Answer: Solutions to the above equation are also solutions to the equation

$$x + \sqrt[3]{x} - 1 = 0.$$

To see that this equation has a solution, let $f(x) = x + \sqrt[3]{x} - 1$. Then

$$f(0) = -1$$
$$f(1) = 1.$$

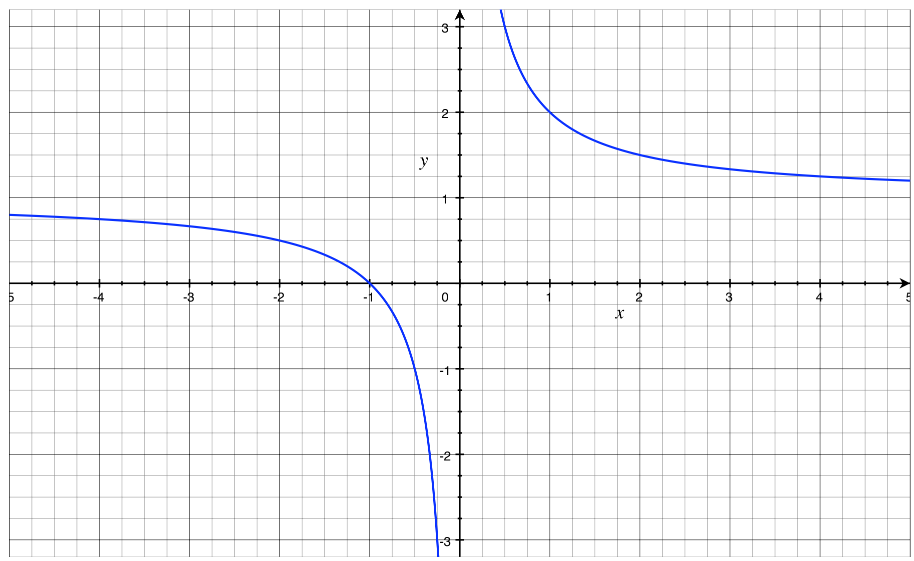
Since $f(x)$ is a continuous function, the Intermediate Value Theorem guarantees that there is some c between 0 and 1 such that $f(c) = 0$...but this c is precisely a solution to the given equation.

§2.6

6. Sketch the graph of an example of a function f that satisfies all of the conditions

$$\lim_{x\to 0^+} f(x) = \infty, \quad \lim_{x\to 0^-} f(x) = -\infty, \quad \lim_{x\to\infty} f(x) = 1, \quad \lim_{x\to -\infty} f(x) = 1.$$

Answer:



18. Find the limit

$$\lim_{y\to\infty} \frac{2 - 3y^2}{5y^2 + 4y}$$

Answer: Dividing both numerator and denominator by y^2, we get

$$\lim_{y\to\infty} \frac{\frac{1}{y^2}\left(2 - 3y^2\right)}{\frac{1}{y^2}\left(5y^2 + 4y\right)} = \lim_{y\to\infty} \frac{\frac{2}{y^2} - 3}{5 + \frac{4}{y}},$$

which, using the Limit Law for quotients, is equal to

$$\frac{\lim_{y\to\infty}\left(\frac{2}{y^2} - 3\right)}{\lim_{y\to\infty}\left(5 + \frac{4}{y}\right)} = -\frac{3}{5}.$$

Therefore,

$$\lim_{y\to\infty} \frac{2-3y^2}{5y^2+4y} = -\frac{3}{5}.$$

28. Find the limit

$$\lim_{x\to\infty} \cos x.$$

Answer: Because $\cos(2n\pi) = 1$ for any n and because $\cos((2n+1)\pi) = -1$ for any n, this limit cannot exist: no matter how far out we go on the x-axis, $\cos x$ is still oscillating between 1 and -1, so it never settles down to a limit.

58. **(a)** A tank contains 5000 L of pure water. Brine that contains 30 g of salt per liter of water is pumped into the tank at a rate of 25 L/min. Show that the concentration of salt after t minutes (in grams per liter) is

$$C(t) = \frac{30t}{200+t}.$$

Answer: Since the tank starts with 5000 L of water and since 25 L/min of brine pours in, the number of liters of water in the tank is given by

$$5000 + 25t.$$

On the other hand, the tank starts with no salt in it and, for each liter of brine that pours in, 30 g of salt pours in. Hence, the number of grams of salt in the tank is given by

$$25(30t).$$

Therefore, the concentration of salt is given by

$$C(t) = \frac{25(30t)}{5000+25t} = \frac{25(30t)}{25(200+t)} = \frac{30t}{200+t}.$$

(b) What happens to the concentration as $t \to \infty$?

Answer: As $t \to \infty$, the concentration of salt is given by

$$\lim_{t\to\infty} C(t) = \lim_{t\to\infty} \frac{30t}{200+t}.$$

Dividing both numerator and denominator by t yields

$$\lim_{t\to\infty} \frac{\frac{1}{t}30t}{\frac{1}{t}(200+t)} = \lim_{t\to\infty} \frac{30}{\frac{200}{t}+1} = 30.$$

So eventually the concentration of salt in the tank approaches 30 g/L, which is the same as the concentration of salt in the brine. In other words, the brine eventually overpowers the pure water.

Extra Credit: Why is problem 58 sort of bogus?

Answer: 58(b) makes it sound like the concentration of salt in an actual tank approaches 30 g/L. But as $t \to \infty$ the amount of water in the tank also goes to infinity. So this limiting behavior could never happen in an actual tank.