Math 113 HW #2 Solutions

§1.6

20. In the theory of relativity, the mass of a particle with speed v is

$$m = f(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the rest mass of the particle and c is the speed of light in a vacuum. Find the inverse function of f and explain its meaning.

Answer: We can find the inverse function by solving for v in the above expression. First, multiply both sides by $\frac{\sqrt{1-v^2/c^2}}{m}$:

$$\sqrt{1 - v^2/c^2} = \frac{m_0}{m}.$$

Squaring both sides yields

$$1 - v^2/c^2 = \frac{m_0^2}{m^2}.$$

This means that

$$v^2 = c^2\left(1 - \frac{m_0^2}{m^2}\right),$$

so we have that

$$v = f^{-1}(m) = c\sqrt{1 - \frac{m_0^2}{m^2}}.$$

If we measure the mass of a moving particle, this expression allows us to determine the velocity of the particle.

26. Find a formula for the inverse of the function

$$y = \frac{e^x}{1 + 2e^x}.$$

Answer: To find the inverse, we first swap the roles of x and y:

$$x = \frac{e^y}{1 + 2e^y}.$$

Now, the goal is to solve for y, so multiply both sides by $1 + 2e^y$:

$$x(1 + 2e^y) = e^y.$$

Subtracting e^y from both sides yields:

$$x + 2xe^y - e^y = 0.$$

To isolate the y's, first subtract x from both sides:

$$e^y(2x - 1) = -x.$$

Now, dividing by $2x-1$, we see that

$$e^y = \frac{-x}{2x-1}.$$

Finally, taking the natural logarithm of both sides yields

$$y = \ln\left(\frac{-x}{2x-1}\right),$$

which is an expression for $f^{-1}(x)$.

36. Find the exact value of each expression:

(a) $e^{-2\ln 5}$

Answer: First, notice that

$$-2\ln 5 = \ln\left(5^{-2}\right) = \ln\left(\frac{1}{5^2}\right) = \ln\left(\frac{1}{25}\right).$$

Therefore, since $e^{\ln x} = x$ for any $x > 0$, we have that

$$e^{-2\ln 5} = e^{\ln\left(\frac{1}{25}\right)} = \frac{1}{25}.$$

(b) $\ln\left(\ln e^{e^{10}}\right)$

Answer: Since $\ln\left(e^x\right) = x$ for any x, we have that

$$\ln e^{e^{10}} = e^{10}.$$

Therefore,

$$\ln\left(\ln e^{e^{10}}\right) = \ln\left(e^{10}\right) = 10.$$

38. Express the quantity

$$\ln(a+b) + \ln(a-b) - 2\ln c$$

as a single logarithm.

Answer: Using the properties of logarithms, we know that

$$\ln(a+b) + \ln(a-b) = \ln\left[(a+b)(a-b)\right] = \ln\left(a^2-b^2\right)$$

and that

$$2\ln c = \ln\left(c^2\right).$$

Hence,

$$\ln(a+b) + \ln(a-b) - 2\ln c = \ln\left(a^2-b^2\right) - \ln\left(c^2\right);$$

in turn, this is equal to

$$\ln\left(\frac{a^2-b^2}{c^2}\right).$$

50. Solve each equation for x.

(a) $\ln(\ln x) = 1$

Answer: Since $e^{\ln x} = x$, we can exponentiate both sides to see that

$$\ln x = e^1 = e.$$

Exponentiating both sides again yields

$$x = e^e.$$

(b) $e^{ax} = Ce^{bx}$, where $a \neq b$.

Answer: Dividing both sides by e^{bx}, we have that

$$\frac{e^{ax}}{e^{bx}} = C.$$

However, the left side can be re-written, using the properties of exponentials, as $e^{ax-bx} = e^{(a-b)x}$, so we have that

$$e^{(a-b)x} = C.$$

Now, taking the natural logarithm of both sides, we have that

$$(a-b)x = \ln C.$$

Dividing both sides by $a - b$ gives the expression for x:

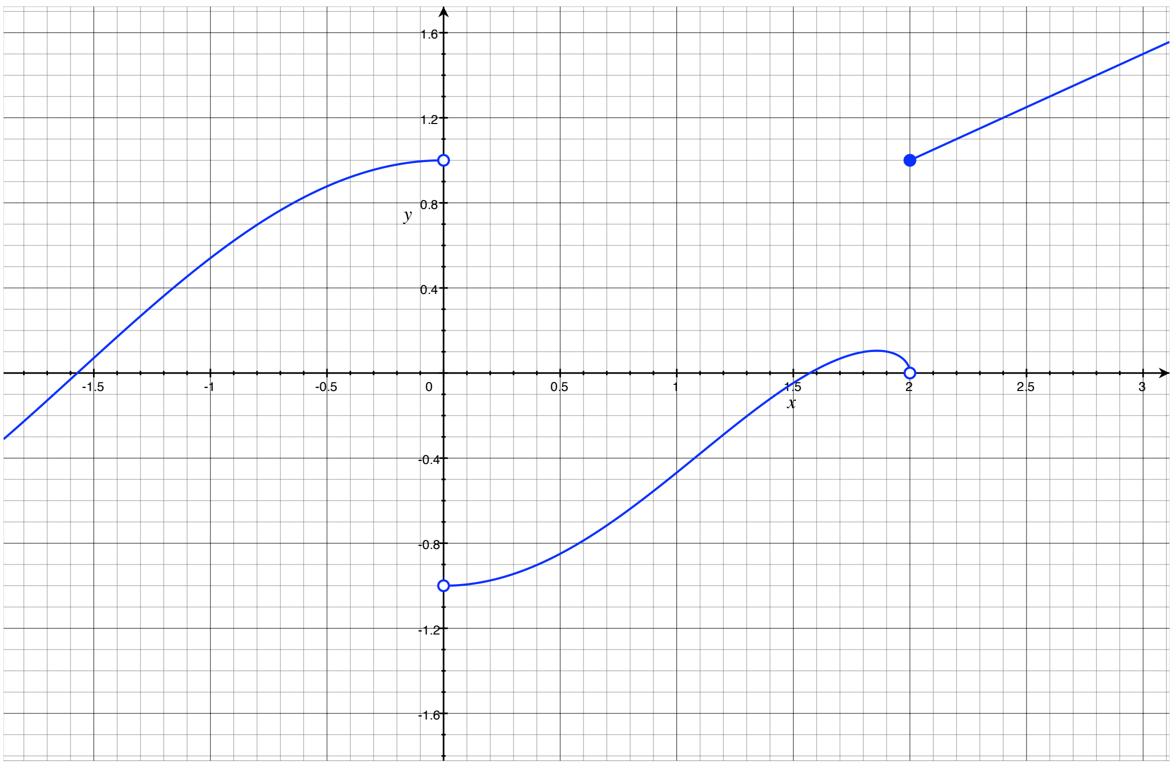
$$x = \frac{\ln C}{a-b}.$$

§2.2

14. Sketch the graph of a function f that satisfies all of the following conditions:

$$\lim_{x\to 0^-} f(x) = 1, \quad \lim_{x\to 0^+} f(x) = -1, \quad \lim_{x\to 2^-} f(x) = 0,$$
$$\lim_{x\to 2^+} f(x) = 1, \quad f(2) = 1, \quad f(0) \text{ is undefined}$$

Answer:



28. Determine the infinite limit

$$\lim_{x\to 5^-} \frac{e^x}{(x-5)^3}.$$

Answer: Whenever $x < 5$, the expression $x - 5$ will be a negative number. Therefore, as $x \to 5^-$, we see that $x - 5$ becomes a very small negative number. Since taking the cube preserves sign and since $e^x > 0$ for all x, this means that

$$\lim_{x\to 5^-} \frac{e^x}{(x-5)^3} = -\infty.$$

40. In the theory of relativity, the mass of a particle with velocity v is given by

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the mass of the particle at rest and c is the speed of light. What happens as $v \to c^-$?

Answer: Whenever $v < c$, the fraction $\frac{v}{c} < 1$. Therefore, as $v \to c^-$, the fraction

$$\frac{v^2}{c^2}$$

gets closer and closer to 1, but is always less than 1. Hence, the quantity

$$1 - \frac{v^2}{c^2}$$

is always positive but approaches zero as $v \to c^-$. Therefore, as $v \to c^-$, the mass

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

approaches $+\infty$.

§2.3

14. If it exists, evaluate the limit

$$\lim_{x \to 4} \frac{x^2 - 4x}{x^2 - 3x - 4}.$$

Answer: Since $x^2 - 4x = x(x-4)$ and since $x^2 - 3x - 4 = (x+1)(x-4)$, we have that

$$\lim_{x \to 4} \frac{x^2 - 4x}{x^2 - 3x - 4} = \lim_{x \to 4} \frac{x(x-4)}{(x+1)(x-4)} = \lim_{x \to 4} \frac{x}{x+1} = \frac{4}{5}.$$

26. If it exists, evaluate the limit

$$\lim_{t \to 0} \left(\frac{1}{t} - \frac{1}{t^2 + t} \right).$$

Answer: In this case, it's helpful to simplify the expression inside the parentheses. Since $t^2 + t = t(t+1)$, we have that

$$\frac{1}{t} - \frac{1}{t^2 + t} = \frac{t+1}{t(t+1)} - \frac{1}{t(t+1)} = \frac{t}{t(t+1)}.$$

Therefore,

$$\lim_{t \to 0} \left(\frac{1}{t} - \frac{1}{t^2 + t} \right) = \lim_{t \to 0} \frac{t}{t(t+1)} = \lim_{t \to 0} \frac{1}{t+1} = 1.$$

38. Prove that $\lim_{x \to 0^+} \sqrt{x} e^{\sin(\pi/x)} = 0$.

Answer: Since $-1 \le \sin(\pi/x) \le 1$ it should be fairly easy to get upper and lower bounds for the expression $\sqrt{x} e^{\sin(\pi/x)}$. If we can do that, then there's a good chance we might be able to use the Squeeze Theorem.

We know that $-1 \leq \sin(\pi/x) \leq 1$ and that e^x is an increasing function, so we get

$$\frac{1}{e} = e^{-1} \leq e^{\sin(\pi/x)} \leq e^1 = e.$$

Therefore,

$$\frac{\sqrt{x}}{e} \leq \sqrt{x}e^{\sin(\pi/x)} \leq e\sqrt{x}.$$

However,

$$\lim_{x\to 0^+} \frac{\sqrt{x}}{e} = \frac{1}{e} \lim_{x\to 0^+} \sqrt{x} = 0$$

and

$$\lim_{x\to 0^+ e\sqrt{x}} = e \lim_{x\to 0^+} \sqrt{x} = 0.$$

Therefore, by the Squeeze Theorem,

$$\lim_{x\to 0^+} \sqrt{x}e^{\sin(\pi/x)} = 0.$$

56. If $\lim_{x\to 0} \frac{f(x)}{x^2} = 5$, find the following limits.

(a) $\lim_{x\to 0} f(x)$

Answer: The only way I can see how to do this is to re-express what we want in terms of what we know. Since we know $\lim_{x\to 0} \frac{f(x)}{x^2}$, it would be nice to express $f(x)$ in terms of $\frac{f(x)}{x^2}$. We can do this when $x \neq 0$:

$$f(x) = \frac{f(x)}{x^2}x^2.$$

Therefore,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \left(\frac{f(x)}{x^2}x^2\right).$$

But, using one of the limit laws, this is equal to

$$\left(\lim_{x\to 0} \frac{f(x)}{x^2}\right)\left(\lim_{x\to 0} x^2\right) = 5 \cdot 0 = 0.$$

(b) $\lim_{x\to 0} \frac{f(x)}{x}$

Answer: When $x \neq 0$,

$$\frac{f(x)}{x} = \frac{f(x)}{x^2}x.$$

Therefore,

$$\lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \left(\frac{f(x)}{x^2}x\right).$$

Using one of the limit laws, this is equal to

$$\left(\lim_{x\to 0} \frac{f(x)}{x^2}\right)\left(\lim_{x\to 0} x\right) = 5 \cdot 0 = 0.$$