

## Math 113 HW #10 Solutions

### §4.5

14. Use the guidelines of this section to sketch the curve

$$y = \frac{x^2}{x^2 + 9}.$$

**Answer:** Using the quotient rule,

$$y' = \frac{(x^2 + 9)(2x) - x^2(2x)}{(x^2 + 9)^2} = \frac{18x}{(x^2 + 9)^2}.$$

Since the denominator is always positive, the sign of  $y'$  is the same as the sign of the numerator. Therefore,  $y' < 0$  when  $x < 0$  and  $y' > 0$  when  $x > 0$ . Hence,  $y$  is decreasing for  $x < 0$ ,  $y$  is increasing for  $x > 0$  and, by the first derivative test,  $y$  has a local minimum of 0 at  $x = 0$ .

Taking the second derivative using the quotient rule,

$$y'' = \frac{(x^2 + 9)^2(18) - 18x(2(x^2 + 9)(2x))}{(x^2 + 9)^4} = 18 \frac{(x^2 + 9)^2(1 - 4x^2)}{(x^2 + 9)^4} = 18 \frac{1 - 4x^2}{(x^2 + 9)^2}.$$

Notice that  $y''$  is positive for  $-\frac{1}{2} < x < \frac{1}{2}$  and  $y''$  is negative for  $x < -\frac{1}{2}$  and  $x > \frac{1}{2}$ . Hence,  $y$  is concave down on  $(-\infty, -1/2)$  and  $(1/2, \infty)$ ,  $y$  is concave up on  $(-1/2, 1/2)$ , and both  $-1/2$  and  $1/2$  are inflection points.

Finally, notice that

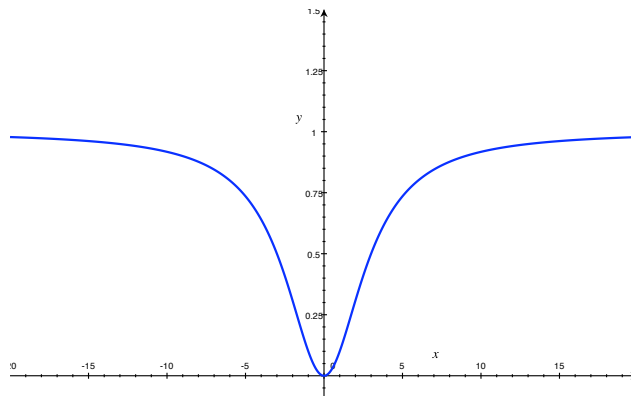
$$\lim_{x \rightarrow \infty} \frac{x^2}{x^2 + 9} = \lim_{x \rightarrow \infty} \frac{1}{1 + 9/x^2} = 1$$

and, likewise

$$\lim_{x \rightarrow -\infty} \frac{x^2}{x^2 + 9} = \lim_{x \rightarrow -\infty} \frac{1}{1 + 9/x^2} = 1,$$

so  $y$  has a horizontal asymptote at  $y = 1$  in both directions.

Putting all the above information together yields a sketch of the curve:



38. Use the guidelines of this section to sketch the curve

$$y = \frac{\sin x}{2 + \cos x}.$$

**Answer:** Using the quotient rule:

$$y' = \frac{(2 + \cos x) \cos x - \sin x(-\sin x)}{(2 + \cos x)^2} = \frac{2 \cos x + \cos^2 x + \sin^2 x}{(2 + \cos x)^2} = \frac{2 \cos x + 1}{(2 + \cos x)^2}.$$

Since the denominator is always non-negative, the sign of  $y'$  is the same as the sign of the numerator,  $2 \cos x + 1$ . Thus,  $y' < 0$  when  $2 \cos x + 1 < 0$ , meaning when

$$\cos x < -\frac{1}{2},$$

which occurs when  $\frac{2\pi}{3} < x < \frac{4\pi}{3}$  or  $\frac{8\pi}{3} < x < \frac{10\pi}{3}$  or, in general, when  $\frac{(6n+2)\pi}{3} < x < \frac{(6n+4)\pi}{3}$  for some integer  $n$ . Therefore,  $y$  is decreasing on the intervals

$$\left( \frac{(6n+2)\pi}{3}, \frac{(6n+4)\pi}{3} \right)$$

and increasing everywhere else. Also, from the first derivative test, we see that  $y$  has local maxima at  $x = \frac{(6n+2)\pi}{3}$  and local minima at  $x = \frac{(6n+4)\pi}{3}$ .

Using the quotient rule again,

$$\begin{aligned} y'' &= \frac{(2 + \cos x)^2(-2 \sin x) - (2 \cos x + 1)(2(2 + \cos x)(-\sin x))}{(2 + \cos x)^4} \\ &= (2 + \cos x) \frac{-4 \sin x - 2 \sin x \cos x + 4 \sin x \cos x + 2 \sin x}{(2 + \cos x)^4} \\ &= \frac{2 \sin x(\cos x - 1)}{(2 + \cos x)^3} \end{aligned}$$

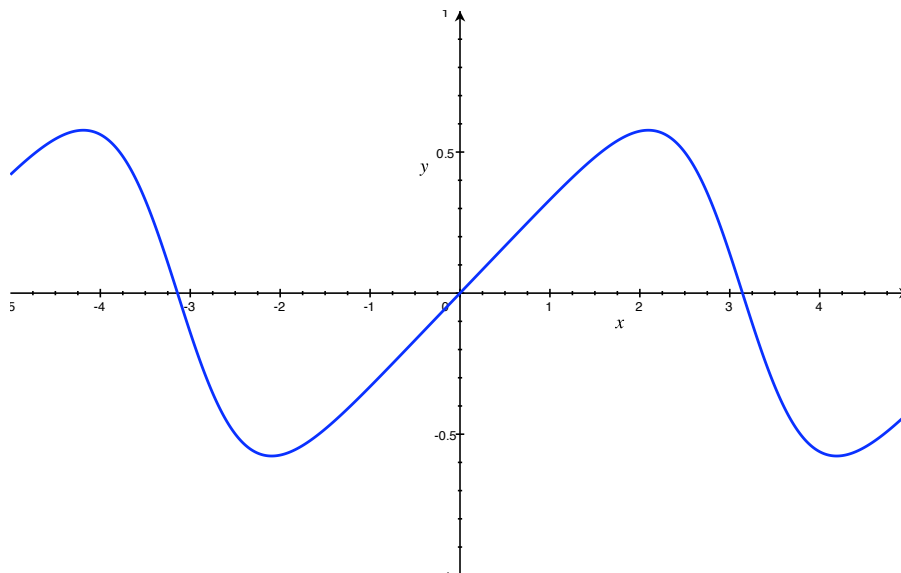
Since the denominator is non-negative, the sign of  $y''$  is the same as the sign of the numerator,  $2 \sin x(\cos x - 1)$ . In turn, since  $\cos x - 1$  is always non-positive, the sign of the numerator is opposite the sign of  $\sin x$ . Therefore,  $y'' < 0$  (meaning  $y$  is concave down) when  $\sin x > 0$ , which happens when  $x$  is above the  $x$ -axis; i.e. when  $2n\pi < x < (2n+1)\pi$  for any integer  $n$ . Likewise,  $y'' > 0$  (and, thus,  $y$  is concave up) when  $x$  is below the  $x$ -axis; that is, when  $(2n+1)\pi < x < (2n+2)\pi$  for any integer  $n$ .

Since  $y''$  changes sign at such points, we see that  $y$  has inflection points at  $x = n\pi$  for any integer  $n$ .

Putting all this together should give a sketch like:

## §4.7

12. A box with a square base and open top must have a volume of  $32,000 \text{ cm}^3$ . Find the dimensions of the box that minimize the amount of material used.



**Answer:** We will use the surface area of the box as a proxy for the amount of material used, so we want to minimize the surface area for the given volume.

To that end, let  $x$  denote the length of the sides on the base, and let  $h$  be the height of the box. Then the volume of the box is given by

$$V = x^2 h.$$

Since the  $V = 32,000$ , we have that

$$32,000 = x^2 h, \quad \text{or} \quad h = \frac{32,000}{x^2}.$$

Now, the surface area of the box is (since it has an open top):

$$A = x^2 + 4xh = x^2 + 4x \frac{32,000}{x^2} = x^2 + \frac{128,000}{x}.$$

In other words, our goal is to minimize the function  $A(x) = x^2 + \frac{128,000}{x}$ .

Note that

$$A'(x) = 2x - \frac{128,000}{x^2},$$

so we have a critical point when

$$0 = 2x - \frac{128,000}{x^2}.$$

Multiplying both sides by  $x^2$  yields

$$0 = 2x^3 - 128,000.$$

Hence,  $x^3 = 64,000$ , meaning that  $x = 40$ .

Note that  $A''(x) = 2 + \frac{256,000}{x^3}$ , so  $A''(40) = 4 > 0$ , so  $x = 40$  is a minimum of the function  $A$ .

Therefore, the box uses the minimum amount of materials when  $x = 40$  and  $h = \frac{32,000}{40^2} = 20$ .

- 30.** A Norman window has the shape of a rectangle surmounted by a semicircle (Thus the diameter of the semicircle is equal to the width of the rectangle. See Exercise 56 on page 23.) If the perimeter of the window is 30 ft, find the dimensions of the window so that the greatest possible amount of light is admitted.

**Answer:** Let  $x$  denote half the width of the rectangle (so  $x$  is the radius of the semicircle), and let  $h$  denote the height of the rectangle. Then the perimeter of the window is

$$2y + 2x + \frac{1}{2}(2\pi x) = 2y + (2 + \pi)x.$$

Since the perimeter is 30, we have that

$$y = \frac{30 - (2 + \pi)x}{2} = 15 - \left(1 + \frac{\pi}{2}\right)x.$$

Therefore, the area of the window (which is proportional to the amount of light admitted), is given by

$$A = (2x)y + \frac{1}{2}(\pi x^2) = 2xy + \frac{\pi x^2}{2}.$$

Substituting the above value for  $y$  yields

$$A(x) = 2x \left[ 15 - \left(1 + \frac{\pi}{2}\right)x \right] + \frac{\pi x^2}{2} = 30x - \left(2 + \frac{\pi}{2}\right)x^2.$$

This is the quantity we're trying to maximize, so take the derivative and find the critical points:

$$A'(x) = 30 - 2\left(2 + \frac{\pi}{2}\right)x = 30 - (4 + \pi)x.$$

Therefore,  $A'(x) = 0$  when  $30 - (4 + \pi)x = 0$  or, equivalently, when

$$x = \frac{30}{4 + \pi} \approx 4.2.$$

Since the domain of  $A$  is  $\left[0, \frac{15}{1 + \pi/2}\right]$  (since both  $x$  and  $y$  must be non-negative), we evaluate  $A$  at the critical point and the endpoints:

$$\begin{aligned} A(0) &= 0 \\ A\left(\frac{30}{4 + \pi}\right) &\approx 63.0 \\ A\left(\frac{15}{1 + \pi/2}\right) &\approx 53.5 \end{aligned}$$

Therefore, the maximum comes at the critical point  $x = \frac{30}{4 + \pi} \approx 4.2$ , which implies the other dimension yielding maximum area is

$$y = 15 - \left(1 + \frac{\pi}{2}\right)\frac{30}{4 + \pi} \approx 4.2.$$

Hence, the window allowing maximal light in is the one with square base.

58. The manager of a 100-unit apartment complex knows from experience that all units will be occupied if the rent is \$800 per month. A market survey suggests that, on average, one additional unit will remain vacant for each \$10 increase in rent. What rent should the manager charge to maximize revenue?

**Answer:** First, we want to determine the price (or demand) function  $p(x)$ . Assuming it is linear, we know that  $y = p(x)$  passes through the point  $(100, 800)$  (corresponding to the building being full when \$800/month is charged), so we just need to determine the slope of the line.

$$\text{slope} = \frac{\Delta \text{price}}{\Delta \text{occupancy}} = \frac{+10}{-1} = -10.$$

Therefore, we want the equation of the line of slope  $-10$  passing through  $(100, 800)$ :

$$y - 800 = -10(x - 100)$$

or, equivalently,

$$y = -10x + 1800.$$

Therefore,  $p(x) = -10x + 1800$ .

Now, revenue equals the price charged (in this case,  $p(x)$ ) times the number of units rented ( $x$ ), so

$$R(x) = xp(x) = x(-10x + 1800) = -10x^2 + 1800x.$$

We want to maximize  $R$ , so we find the critical points:

$$R'(x) = -20x + 1800,$$

so  $R'(x) = 0$  when

$$0 = -20x + 1800.$$

Therefore, the single critical point occurs when  $x = \frac{1800}{20} = 90$ . Since  $R''(x) = -20$ , we see that  $R$  is concave down everywhere and so the absolute maximum of the function  $R$  must occur at this critical point.

This says that the manager maximizes his revenue when he has 90 tenants, which means he ought to charge

$$p(90) = -10(90) + 1800 = -900 + 1800 = \$900 \text{ per month}$$

to rent a unit.

68. A rain gutter is to be constructed from a metal sheet of width 30 cm by bending up one-third of the sheet on each side through an angle  $\theta$ . How should  $\theta$  be chosen so that the gutter will carry the maximum amount of water?

**Answer:** The amount of water that the gutter can carry is proportional to the area of a cross section of the gutter. If  $h$  is the height that the tip of one of the bent-up segments rises above the base, then the cross-sectional area is

$$A = 10h + 2 \left( \frac{1}{2}bh \right).$$

Now, by the definition of the sine,

$$\sin \theta = \frac{h}{10}, \quad \text{so} \quad h = 10 \sin \theta.$$

Likewise,

$$\cos \theta = \frac{b}{10}, \quad \text{so} \quad b = 10 \cos \theta.$$

Therefore,

$$A(\theta) = 10(10 \sin \theta) + (10 \sin \theta)(10 \cos \theta) = 100 \sin \theta + 100 \sin \theta \cos \theta.$$

This is the function we're trying to maximize, so differentiate and find critical points:

$$A'(\theta) = 100 \cos \theta + 100(\sin \theta(-\sin \theta) + \cos \theta(\cos \theta)) = 100(\cos \theta - \sin^2 \theta + \cos^2 \theta).$$

Using the fact that  $\sin^2 \theta + \cos^2 \theta = 1$ , we can write  $\sin^2 \theta = 1 - \cos^2 \theta$ , so

$$A'(\theta) = 100(\cos \theta - (1 - \cos^2 \theta) + \cos^2 \theta) = 100(2 \cos^2 \theta + \cos \theta - 1) = 100(2 \cos \theta - 1)(\cos \theta + 1).$$

Therefore,  $A'(\theta) = 0$  when

$$100(2 \cos \theta - 1)(\cos \theta + 1) = 0$$

or, equivalently, when either

$$\cos \theta = \frac{1}{2} \quad \text{or} \quad \cos \theta = -1.$$

Thus,  $\theta = \frac{\pi}{3}$  or  $\theta = \pi$ . Now,  $\theta$  can range from 0 to  $\pi$ , so we just plug in critical points and endpoints into the area function:

$$\begin{aligned} A(0) &= 0 \\ A(\pi/3) &= 100 \frac{\sqrt{3}}{2} + 100 \frac{\sqrt{3}}{2} \frac{1}{2} = \frac{300\sqrt{3}}{4} \approx 129.9 \\ A(\pi) &= 0. \end{aligned}$$

Therefore, the gutter can carry the maximum amount of water when  $\theta = \pi/3$ .

## §4.8

16. Use Newton's method to approximate the positive root of  $2 \cos x = x^4$  correct to six decimal places.

**Answer:** Let  $f(x) = 2 \cos x - x^4$ . Then we want to use Newton's method to find the  $x > 0$  such that  $f(x) = 0$ .

Notice that

$$f'(x) = -2 \sin x - 4x^3.$$

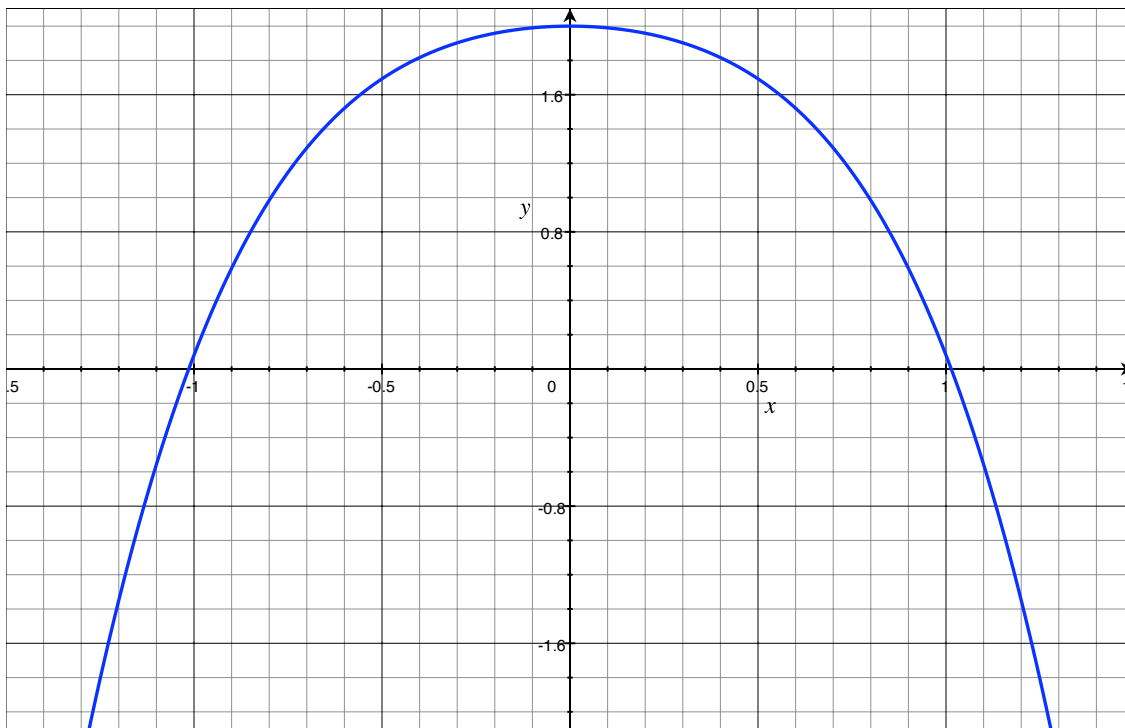


Figure 1:  $f(x) = 2 \cos x - x^4$

Now, based on the graph of  $f$ , guess  $x_0 = 1$ . Then

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{2 \cos(1) - 1^4}{-2 \sin(1) - 4(1)^3} \approx 1.014183.$$

Then

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 1.014183 - \frac{2 \cos(1.014183) - 1.014183^4}{-2 \sin(1.014183) - 4(1.014183)^3} \approx 1.013957.$$

In turn,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 1.013957 - \frac{2 \cos(1.013957) - 1.013957^4}{-2 \sin(1.013957) - 4(1.013957)^3} \approx 1.013957,$$

so we can stop, since this is the same as  $x_2$  to six decimal places. Therefore, the positive root of the equation  $2 \cos x = x^4$  is, to six decimal places, 1.013957.

- 26.** Use Newton's method to find all the roots of the equation  $3 \sin(x^2) = 2x$  correct to eight decimal places. Start by drawing a graph to find initial approximations.

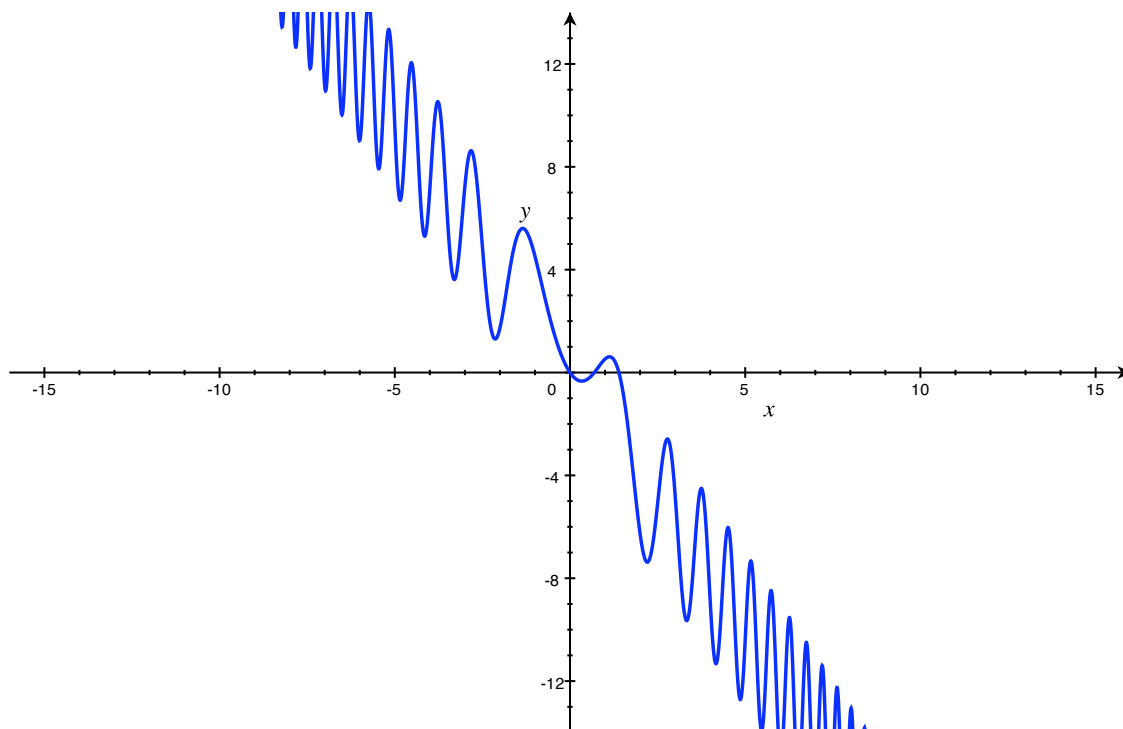
**Answer:** Let  $f(x) = 3 \sin(x^2) - 2x$ . We want to approximate the values of  $x$  such that  $f(x) = 0$ . We'll need to use the derivative of  $f$ , so compute

$$f'(x) = 3 \cos(x^2) \cdot 2x - 2 = 6x \cos(x^2) - 2.$$

In general,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3 \sin(x_n^2) - 2x_n}{6x_n \cos(x_n^2) - 2}.$$

Now, the graph of  $f$  looks like:



From the figure, it appears there are three zeroes of  $f(x)$ . One seems to be at the origin and, in fact,

$$f(0) = 3 \sin(0^2) - 2(0) = 0 - 0 = 0,$$

so one root of the equation is zero.

The next root is approximately  $1/2$ , so guess  $x_0 = 1/2$  and use Newton's Method to compute the following sequence:

$$\begin{aligned} x_0 &= \frac{1}{2} \\ x_1 &\approx 0.78430299 \\ x_2 &\approx 0.69609320 \\ x_3 &\approx 0.69300735 \\ x_4 &\approx 0.69299995 \\ x_5 &\approx 0.69299995 \end{aligned}$$

We can stop here and conclude that, to eight decimal places, the second root of the equation is 0.69299995.



Based on the graph, the last root of  $f$  is approximately  $3/2$ , so start Newton's Method with the guess  $x_0 = 3/2$ :

$$\begin{aligned}x_0 &= \frac{3}{2} \\x_1 &\approx 1.41301039 \\x_2 &\approx 1.39594392 \\x_3 &\approx 1.39525190 \\x_4 &\approx 1.39525077 \\x_5 &\approx 1.39525077\end{aligned}$$

Thus the third root of the equation is, to eight decimal places, 1.39525077.

Putting it all together, we see that, with eight decimal places' accuracy, the three roots of the equation  $3 \sin(x^2) = 2x$  are

$$0, 0.69299995, 1.39525077.$$

- 30. (a)** Apply Newton's method to the equation  $1/x - a = 0$  to derive the following reciprocal algorithm:

$$x_{n+1} = 2x_n - ax_n^2.$$

(This enables a computer to find reciprocals without actually dividing.)

**Answer:** Let  $f(x) = 1/x - a$ . Then the derivative of  $f$  is given by

$$f'(x) = -\frac{1}{x^2},$$

so the appropriate sequence for Newton's Method is determined by the recurrence relation

$$\begin{aligned}x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\&= x_n - \frac{\frac{1}{x_n} - a}{-\frac{1}{x_n^2}} \\&= x_n - (-x_n^2) \left( \frac{1}{x_n} - a \right) \\&= x_n + x_n - ax_n^2 \\&= 2x_n - ax_n^2,\end{aligned}$$

as desired.

- (b)** Use part (a) to compute  $1/1.6984$  correct to six decimal places.

**Answer:** Let  $a = 1.6984$  in the above expression, so

$$x_{n+1} = 2x_n - 1.6984x_n^2.$$

Now, since 1.6984 is a bit smaller than 2,  $1/1.6984$  should be a little bigger than  $1/2$ , so  $x_0 = 1/2$  isn't a bad guess. Then using the above expression to compute  $x_1, x_2, \dots$ , we get the sequence

$$\begin{aligned}x_0 &= \frac{1}{2} \\x_1 &= 0.5754 \\x_2 &\approx 0.588484 \\x_3 &\approx 0.588789 \\x_4 &\approx 0.588789\end{aligned}$$

so, to six decimal places' accuracy,  $\frac{1}{1.6984} \approx 0.588789$ .

## §4.9

10. Find the most general antiderivative of the function

$$f(x) = \sqrt[4]{x^3} + \sqrt[3]{x^4}.$$

**Answer:** We can re-write  $f$  as

$$f(x) = x^{3/4} + x^{4/3}.$$

Then, using the reverse of the power rule, it's easy to see that the following is the most general antiderivative of  $f$ :

$$\frac{x^{7/4}}{7/4} + \frac{x^{7/3}}{7/3} + C = \frac{4}{7}\sqrt[4]{x^7} + \frac{3}{7}\sqrt[3]{x^7} + C.$$

44. Find  $f$  given that

$$f''(t) = 2e^t + 3\sin t, \quad f(0) = 0, \quad f(\pi) = 0.$$

**Answer:** Given what we know about  $f''(t)$ , we can determine  $f'(t)$  as follows:

$$f'(t) = \int f''(t)dt = \int (2e^t + 3\sin t)dt = 2e^t - 3\cos t + C.$$

In turn, this means that

$$f(t) = \int f'(t)dt = \int (2e^t - 3\cos t + C)dt = 2e^t - 3\sin t + Ct + D,$$

where  $C$  and  $D$  are both (as yet unknown) constants. To determine  $C$  and  $D$ , we use our knowledge of the values of  $f$ . Plugging in  $t = 0$ , we have that

$$0 = f(0) = 2e^0 - 3\sin(0) + C(0) + D = 2 - 0 + 0 + D = 2 + D,$$

so it must be the case that  $D = -2$ . Hence,

$$f(t) = 2e^t - 3\sin t + Ct - 2.$$

Now, plugging in  $t = \pi$ , we have that

$$0 = f(\pi) = 2e^\pi - 3\sin(\pi) + C\pi - 2 = 2e^\pi - 0 + C\pi - 2 = 2e^\pi + C\pi - 2.$$

Therefore,  $C = \frac{2-2e^\pi}{\pi}$ , so we conclude that

$$f(t) = 2e^t - 3\sin t + \frac{2-2e^\pi}{\pi}t - 2.$$

50. The graph of a function  $f$  is shown. Which graph is an antiderivative of  $f$  and why?

**Answer:** The only graph which can be an antiderivative of  $f$  is  $a$ . To see this, note that, when  $f$  is positive, its antiderivative should be increasing, which eliminates  $b$  from consideration. Also, when  $f$  is negative, its antiderivative should be decreasing; this eliminates  $c$ , which is increasing for all visible  $x$ .

60. A particle is moving with the data

$$a(t) = \cos t + \sin t, \quad s(0) = 0, \quad v(0) = 5.$$

Find the position (i.e.  $s(t)$ ) of the particle

**Answer:** Since  $a(t) = v'(t)$ , we can determine  $v$  as follows:

$$v(t) = \int a(t)dt = \int (\cos t + \sin t)dt = \sin t - \cos t + C.$$

Now, since  $v(0) = 5$ , we can plug in  $t = 0$  to see that

$$5 = v(0) = \sin(0) - \cos(0) + C = 0 - 1 + C = C - 1,$$

so  $C = 6$  and we have that

$$v(t) = \sin t - \cos t + 6.$$

Now, since  $v(t) = s'(t)$ , we have that

$$s(t) = \int v(t)dt = \int (\sin t - \cos t + 6)dt = -\cos t - \sin t + 6t + D.$$

Now, plugging in  $t = 0$ , we have that

$$0 = s(0) = -\cos(0) - \sin(0) + 6(0) + D = -1 - 0 - 0 + D = D - 1,$$

so  $D = 1$ . Therefore, the position of the particle is given by

$$s(t) = -\cos t - \sin t + 6t + 1.$$