PFP MATH 104 QUIZ II SOLUTIONS

(1) Express

$$\int_0^2 (x^2 - 3x)dx$$

as a limit of Riemann sums (but don't evaluate this limit). Then use the Fundamental Theorem of Calculus to solve this definite integral.

Answer: Let $f(x) = x^2 - 3x$. We approximate this definite integral with the area of n thin strips using the right endpoint. To do so, we need to break the interval $[0, 2]$ up into n subintervals $[x_i, x_{i+1}]$, so the width of each subinterval is

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}.$$

Note that this implies that $x_i = \frac{2}{n}i$. Now, the height of the ith strip is

$$f(x_i) = f\left(\frac{2}{n}i\right) = \left(\frac{2}{n}i\right)^2 - 3\left(\frac{2}{n}i\right) = \frac{4i^2}{n^2} - \frac{6i}{n}.$$

Therefore, letting $n \to \infty$,

$$\begin{aligned}\int_0^2 (x^2 - 3x)dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(x_i) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2}{n}\right)\left(\frac{4i^2}{n^2} - \frac{6i}{n}\right) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n^2}\left(\frac{2i^2}{n} - 3i\right) \\ &= \lim_{n\to\infty} \frac{4}{n^2} \sum_{i=1}^{n} \left(\frac{2i^2}{n} - 3i\right).\end{aligned}$$

Now, using the Fundamental Theorem of Calculus, Part II to evaluate this integral,

$$\int_0^2 (x^2 - 3x)dx = \left[\frac{x^3}{3} - 3\frac{x^2}{2}\right]_0^2 = \frac{8}{3} - 6 = \frac{10}{3}.$$

(2) Suppose

$$\int_1^x f(t)dt = x \cos \pi x.$$

What is $f(2)$?

Answer: Let $g(x) = x\cos\pi x$. Using the Fundamental Theorem of Calculus, Part I,

$$g'(x) = f(x).$$

In particular, $f(2) = g'(2)$. Therefore, we just need to differentiate g and evaluate the result at $x = 2$. To that end,

$$g'(x) = 1 \cdot \cos\pi x + x\left(-\pi\sin\pi x\right) = \cos\pi x - \pi x\sin\pi x,$$

using the product rule. Therefore,

$$f(2) = g'(2) = \cos 2\pi - 2\pi\sin 2\pi = 1 - 0 = 1.$$

(3) Evaluate the integral

$$\int_1^8 \frac{t-1}{\sqrt[3]{t^2}}\,dt.$$

Answer: We can re-write the integral as

$$\int_1^8 \frac{t-1}{\sqrt[3]{t^2}}\,dt = \int_1^8 \left(\frac{t}{\sqrt[3]{t^2}} - \frac{1}{\sqrt[3]{t^2}}\right)dt = \int_1^8 \left(t^{1/3} - t^{-2/3}\right)dt.$$

Therefore,

$$\begin{aligned}\int_1^8 \frac{t-1}{\sqrt[3]{t^2}}\,dt = \int_1^8 \left(t^{1/3} - t^{-2/3}\right)dt &= \left[\frac{3}{4}t^{4/3} - 3t^{1/3}\right]_1^8 \\ &= \left(\frac{3}{4}\cdot 16 - 3\cdot 2\right) - \left(\frac{3}{4}\cdot 1 - 3\right) \\ &= 9 - \frac{3}{4} \\ &= \frac{33}{4}.\end{aligned}$$

(4) Suppose the velocity of a particle at time t is given by the expression

$$v(t) = \frac{\sin t + \cos t}{2}$$

for $t \geq 0$. How far from its starting point is the particle at time $t = \frac{3\pi}{4}$?

Answer: If $s(t)$ denotes the position function of the particle, then $v(t) = s'(t)$. Therefore, by the Fundamental Theorem of Calculus, Part II, the net change in position is given by

$$\begin{aligned}
s\left(\frac{3\pi}{4}\right) - s(0) = \int_0^{\frac{3\pi}{4}} s'(t)dt &= \int_0^{\frac{3\pi}{4}} v(t)dt \\
&= \int_0^{\frac{3\pi}{4}} \frac{\sin t + \cos t}{2} dt \\
&= \left[\frac{-\cos t}{2} + \frac{\sin t}{2}\right]_0^{\frac{3\pi}{4}} \\
&= \left(\frac{-\cos\left(\frac{3\pi}{4}\right)}{2} + \frac{\sin\left(\frac{3\pi}{4}\right)}{2}\right) - \left(\frac{-\cos 0}{2} + \frac{\sin 0}{2}\right) \\
&= \left(-\left(-\frac{\sqrt{2}}{4}\right) + \frac{\sqrt{2}}{4}\right) - \left(-\frac{1}{2} + 0\right) \\
&= \frac{\sqrt{2}}{2} + \frac{1}{2}.
\end{aligned}$$