PFP MATH 104 QUIZ I SOLUTIONS

(1) Let

$$f(x) = x \cos x.$$

What is the equation of the tangent line to the curve $y = f(x)$ at the point $x = \frac{\pi}{2}$?

Answer: Using the product rule,

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x) \cos x + x \frac{d}{dx}(\cos x) \\ &= 1 \cdot \cos x + x(-\sin x) \\ &= \cos x - x \sin x. \end{aligned}$$

Now, when we plug in the point $x = \frac{\pi}{2}$, we see that

$$f'\left(\frac{\pi}{2}\right) = 0 - \frac{\pi}{2}(1) = -\frac{\pi}{2},$$

which is the slope of the tangent line.

Since $f\left(\frac{\pi}{2}\right) = 0$, we want a line with slope $-\frac{\pi}{2}$ passing through the point $\left(\frac{\pi}{2}, 0\right)$, so the equation of the line is

$$y - 0 = -\frac{\pi}{2}\left(x - \frac{\pi}{2}\right),$$

which simplifies to

$$y = -\frac{\pi}{2}x + \frac{\pi^2}{4}.$$

(2) A population of 500 bacteria is introduced into a culture and grows in number according to the equation

$$P(t) = 500\left(1 + \frac{4t}{50 + t^2}\right)$$

where t is measured in hours. Find an expression for the rate of population growth of the bacteria.

Answer: The rate at which the population is growing is given by the derivative, $P'(t)$. Since the derivative of a constant is zero,

$$P'(t) = \frac{d}{dt}\left[500\left(1 + \frac{4t}{50 + t^2}\right)\right] = 500 \frac{d}{dt}\left(\frac{4t}{50 + t^2}\right)$$

Using the quotient rule, then, we see that

$$\begin{aligned} P'(t) &= 500\frac{(50+t^2)\frac{d}{dt}(4t) - 4t\frac{d}{dt}(50+t^2)}{(50+t^2)^2} \\ &= 500\frac{(50+t^2)\cdot 4 - 4t\cdot 2t}{(50+t^2)^2} \\ &= 500\frac{200-4t^2}{(50+t^2)^2}. \end{aligned}$$

Therefore, the rate of population growth is given by the expression

$$500\frac{200-4t^2}{(50+t^2)^2}$$

(3) Let

$$g(t) = \frac{\sec t}{t}.$$

What is $g'(\pi)$?

Answer: Using the quotient rule,

$$\begin{aligned} g'(t) &= \frac{t\frac{d}{dt}(\sec t) - \sec t\frac{d}{dt}(t)}{t^2} \\ &= \frac{t\sec t\tan t - \sec t}{t^2} \\ &= \frac{\sec t(t\tan t - 1)}{t^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} g'(\pi) &= \frac{\sec\pi(\pi\tan\pi - 1)}{\pi^2} \\ &= \frac{1}{\pi^2}. \end{aligned}$$

(4) Let

$$y = \sin\sqrt{x} + \sqrt{\sin x}.$$

What is $\frac{dy}{dx}$?

Answer: Let $u = \sqrt{x}$ and let $v = \sin x$. Then we can write y as

$$y = \sin u + \sqrt{v}.$$

Therefore,

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{du}(\sin u)\frac{du}{dx} + \frac{d}{dv}\left(\sqrt{v}\right)\frac{dv}{dx} \\ &= \cos u\left(\frac{1}{2}x^{-1/2}\right) + \left(\frac{1}{2}v^{-1/2}\right)\cos x \\ &= \frac{\cos\sqrt{x}}{2\sqrt{x}} + \frac{\cos x}{2\sqrt{\sin x}}. \end{aligned}$$