PFP MATH 104 FINAL EXAM SOLUTIONS

(1) Evaluate the definite integral

$$\int_1^3 \frac{x^4+1}{x^2} dx.$$

Answer: Simplify the integral as

$$\int_1^3 \left(\frac{x^4}{x^2} + \frac{1}{x^2}\right) dx = \int_1^3 x^2 dx + \int_1^3 \frac{1}{x^2} dx.$$

Therefore, the integral is given by

$$\left[\frac{x^3}{3}\right]_1^3 + \left[-\frac{1}{x}\right]_1^3 = \left(9 - \frac{1}{3}\right) + \left(-\frac{1}{3} + 1\right) = \frac{28}{3}.$$

(2) Let $f(x) = x^{\cos x}$. What is $f'(\pi/2)$?
Answer: By definition

$$x^{\cos x} = e^{\cos x \ln x}.$$

Therefore,

$$\begin{aligned} \frac{d}{dx}\left(x^{\cos x}\right) &= \frac{d}{dx}\left(e^{\cos x \ln x}\right) \\ &= e^{\cos x \ln x}\left(-\sin x \ln x + \frac{\cos x}{x}\right) \\ &= -x^{\cos x} \sin x \ln x + \frac{x^{\cos x} \cos x}{x}. \end{aligned}$$

So we conclude that

$$f'(\pi/2) = -\ln \pi/2.$$

(3) Assuming $x > 3$, evaluate the integral

$$\int \frac{x}{x^2 - 6x + 9} dx.$$

Answer: First, note that $x^2 - 6x + 9 = (x-3)^2$, so we can re-write the integral as

$$\int \frac{x}{(x-3)^2} dx.$$

Now, let $u = x - 3$. Then $x = u + 3$ and $du = dx$, so we can re-write this integral as

$$\begin{aligned}\int \frac{u+3}{u^2}du &= \int \frac{1}{u}du + \int \frac{3}{u^2}du \\ &= \ln|u| - \frac{3}{u} + C \\ &= \ln|x-3| - \frac{3}{x-3} + C.\end{aligned}$$

(4) Define the function $g(x)$ by the equation

$$g(x) = \int_1^{x^2} \frac{\sin t}{\sqrt{t}}dt.$$

What is the derivative of g?

Answer: Using the Chain Rule and the Fundamental Theorem of Calculus,

$$g'(x) = \frac{\sin x^2}{\sqrt{x^2}}\frac{d}{dx}\left(x^2\right) = \frac{\sin x^2}{x} \cdot 2x = 2\sin x^2.$$

(5) Use right endpoints and 3 subintervals to estimate

$$\int_1^4 \frac{1}{x}dx,$$

then write this definite integral as a limit of Riemann sums.

Answer: Let $f(x) = \frac{1}{x}$. If we break the interval $[1, 4]$ up into 3 equal subintervals, $[1, 2]$, $[2, 3]$ and $[3, 4]$, then the right endpoints are given by $x_1 = 2$, $x_2 = 3$ and $x_3 = 4$. Thus, the definite integral is approximated by

$$1 \cdot f(2) + 1 \cdot f(3) + 1 \cdot f(4) = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}.$$

Now, if we break the interval up into n subintervals, each will be of width $\Delta x = \frac{4-1}{n} = \frac{3}{n}$. Also, since x_i is the right endpoint of the ith subinterval, $x_i = 1 + i\Delta x = 1 + i\frac{3}{n}$.

Therefore,

$$\begin{aligned}\int_1^4 \frac{1}{x}dx &= \lim_{n\to\infty}\sum_{i=1}^{n} \Delta x f(x_i) \\ &= \lim_{n\to\infty}\sum_{i=1}^{n} \frac{3}{n}\frac{1}{1+i\frac{3}{n}} \\ &= \lim_{n\to\infty}\sum_{i=1}^{n} \frac{3}{n}\frac{n}{n+3i} \\ &= \lim_{n\to\infty}\sum_{i=1}^{n} \frac{3}{n+3i}.\end{aligned}$$

(6) Find the inverse of the function

$$f(x) = 1000(1 + 0.07)^x.$$

Answer: Switching the roles of x and y, we get

$$x = 1000(1.07)^y.$$

Dividing everything by 1000, we see that

$$\frac{x}{1000} = 1.07^y.$$

Now, take the natural logarithm of both sides:

$$\ln\left(\frac{x}{1000}\right) = \ln\left(1.07^y\right).$$

Using the properties of logarithms, we can re-write this as

$$\ln x - \ln 1000 = y \ln 1.07.$$

Therefore,

$$f^{-1}(x) = y = \frac{\ln x - \ln 1000}{\ln 1.07}.$$

(7) Suppose the velocity of a particle is given by

$$v(t) = 4(t-3)^3 + 7(t-3) - 4.$$

How far is the particle at time $t = 6$ from its starting point (at $t = 0$)?

Answer: Let $s(t)$ denote the position of the particle. Then $s'(t) = v(t)$, so the displacement of the particle is given by

$$s(6) - s(0) = \int_0^6 v(t)dt = \int_0^6 \left[4(t-3)^3 + 7(t-3) - 4\right] dt.$$

Let $u = t - 3$. Then $du = dt$, so we can re-write the above integral as

$$\int_{-3}^{3} \left[4u^3 + 7u - 4\right] du$$

Now, $4u^3$ and $7u$ are odd functions, so, since we're integrating from -3 to 3, these terms integrate to zero. Therefore, the displacement is given by

$$\int_{-3}^{3} -4du = -4u\Big]_{-3}^{3} = -24.$$

(8) Evaluate the definite integral

$$\int_0^1 \frac{dx}{\sqrt[3]{x}\left(x^{2/3}+1\right)}.$$

Answer: Let $u = x^{2/3} + 1$. Then $du = \frac{2}{3}x^{-1/3}dx$, so we can re-write the integral as

$$\frac{3}{2}\int_1^2 \frac{du}{u} = \frac{3}{2}\Big[\ln|u|\Big]_1^2 = \frac{3}{2}\ln 2.$$

(9) What is

$$\frac{d}{dx}\left(e^{e^{e^x}}\right)?$$

Answer: Determining this derivative requires repeated applications of the Chain Rule:

$$\frac{d}{dx}\left(e^{e^{e^x}}\right) = e^{e^{e^x}}\frac{d}{dx}\left(e^{e^x}\right) = e^{e^{e^x}}e^{e^x}\frac{d}{dx}\left(e^x\right) = e^{e^{e^x}}e^{e^x}e^x.$$

After re-ordering we see that

$$\frac{d}{dx}\left(e^{e^{e^x}}\right) = e^x e^{e^x} e^{e^{e^x}}.$$

(10) Find an antiderivative of the function

$$f(x) = \frac{x^3}{\sqrt{x^2+1}}.$$

Answer: Finding an antiderivative is the same thing as evaluating the indefinite integral

$$\int \frac{x^3}{\sqrt{x^2+1}}dx.$$

To that end, let $u = x^2+1$. Then $du = 2xdx$ and we can re-write the integral as

$$\begin{aligned}\frac{1}{2}\int\frac{x^2}{\sqrt{u}}du &= \frac{1}{2}\int\frac{u-1}{\sqrt{u}}du \\ &= \frac{1}{2}\int\left[\sqrt{u}-\frac{1}{\sqrt{u}}\right]du \\ &= \frac{1}{2}\left[\frac{2}{3}u^{3/2}-2\sqrt{u}\right]+C \\ &= \frac{(x^2+1)^{3/2}}{3}-\sqrt{x^2+1}+C.\end{aligned}$$

Therefore,

$$\frac{(x^2+1)^{3/2}}{3}-\sqrt{x^2+1}$$

is an antiderivative of f.

(11) Evaluate the definite integral

$$\int_1^{100}\frac{(\log_{10} x)^3}{x\ln 10}dx.$$

Answer: Let $u = \log_{10} x$. Then $du = \frac{dx}{x\ln 10}$, so we can re-write the integral as

$$\int_0^2 u^3 du = \left.\frac{u^4}{4}\right]_0^2 = 4.$$

(12) Let

$$f(x) = \sqrt{x^3 + x^2 + x + 1},$$

which is well-defined for $x > -1$. What is $(f^{-1})'(2)$?

Answer: Note that $f(1) = 2$, so $f^{-1}(2) = 1$. Then

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)}.$$

Now,

$$\begin{aligned} f'(x) &= \frac{1}{2}(x^3 + x^2 + x + 1)^{-1/2}\frac{d}{dx}(x^3 + x^2 + x + 1) \\ &= \frac{3x^2 + 2x + 1}{2\sqrt{x^3 + x^2 + x + 1}}, \end{aligned}$$

so

$$f'(1) = \frac{3 + 2 + 1}{2\sqrt{4}} = \frac{6}{4} = \frac{3}{2}.$$

Plugging this into our expression for $(f^{-1})'(1)$, we see that

$$(f^{-1})'(1) = \frac{1}{\frac{3}{2}} = \frac{2}{3}.$$