MATH 114 MIDTERM 3 SOLUTIONS

CLAY SHONKWILER

True/False

(1) In Cartesian coordinates, if R is the rectangle $a \leq x \leq b$, $c \leq y \leq d$ in the xy-plane, $\int_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx$.

Answer: True. This is simply Fubini's Theorem.

♣

(2) $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$ has exactly one saddle point.

Answer: False. In fact, f has two saddle points. To see this, we compute critical points:

$$f_x = 6xy - 6x$$
$$f_y = 3x^2 + 3y^2 - 6y$$

When $0 = f_x = 6xy - 6x = 6x(y-1)$, so either $x = 0$ or $y = 1$. If $x = 0$ and $f_y = 0$, then $0 = 3y^2 - 6y = 3y(y-2)$, so either $y = 0$ or $y = 2$. If $y = 1$ and $f_y = 0$, then $0 = 3x^2 + 3 - 6 = 3x^2 - 3$, so $x^2 = 1$, meaning $x = \pm 1$. Therefore, we have the following four critical points for f:

$$(0,0) \quad (0,2) \quad (1,1) \quad (-1,1).$$

Now, to use the second derivative test, we need to compute the second partials:

$$f_{xx} = 6y - 6$$
$$f_{xy} = 6x$$
$$f_{yy} = 6y - 6$$

Hence,

$$f_{xx}(1,1)f_{yy}(1,1) - (f_{xy}(1,1))^2 = 0 - 36 = -36 < 0$$

and

$$f_{xx}(-1,1)f_{yy}(-1,1) - (f_{xy}(-1,1))^2 = 0 - 36 = -36 < 0,$$

so $(1,1)$ and $(-1,1)$ are both saddle points of f.

♣

(3) If $\nabla g(x_0, y_0) = \mathbf{0}$, then the point (x_0, y_0) is a critical point of $g(x, y)$.

Answer: True. Remember, by definition,

$$\nabla g = g_x \mathbf{i} + g_y \mathbf{j},$$

so if $\nabla g(x_0, y_0) = \mathbf{0}$, then $g_x(x_0, y_0) = 0$ and $g_y(x_0, y_0) = 0$, which is precisely what it means for (x_0, y_0) to be a critical point of g.

♣

(4) If R is the rectangle $-a \leq x \leq a$, $-c \leq y \leq c$ in the xy-plane, then

$$\int_{-c}^{c} \int_{-a}^{a} f(x, y) dx dy = 2 \int_{0}^{c} \int_{0}^{a} f(x, y) dx dy.$$

Answer: False. This is only true when f is symmetric about both the x- and y-axes. To see an example where equality does not hold, suppose $f(x, y) = x + y$. Then

$$\begin{aligned}
\int_{-c}^{c} \int_{-a}^{a} f(x, y) dx dy &= \int_{-c}^{c} \int_{-a}^{a} (x + y) dx dy \\
&= \int_{-c}^{c} \left[\frac{x^2}{2} + xy \right]_{-a}^{a} dy \\
&= \int_{-c}^{c} 2ay dy \\
&= \left[ay^2 \right]_{-c}^{c} \\
&= 0.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
2 \int_{0}^{c} \int_{0}^{a} f(x, y) dx dy &= 2 \int_{0}^{c} \int_{0}^{a} (x + y) dx dy \\
&= 2 \int_{0}^{c} \left[\frac{x^2}{2} + xy \right]_{0}^{a} dy \\
&= 2 \int_{0}^{c} \left[\frac{a^2}{2} + ay \right] dy \\
&= 2 \left[\frac{a^2}{2} y + \frac{a^2}{2} y^2 \right]_{0}^{c} \\
&= ca^2 + c^2 a^2,
\end{aligned}$$

which is clearly non-zero for any interesting choices of a and c.

♣

(5) If (a, b) is a critical point of $f(x, y)$ and $f_{xx}(a, b) f_{yy}(a, b) < f_{xy}^2(a, b)$, then f has a saddle point at (a, b).

Answer: True. This is essentially what the second derivative test says. In particular, if the above inequality holds, then

$$f_{xx}(a, b) f_{yy}(a, b) - f_{xy}^2(a, b) < 0,$$

so (a, b) is a saddle point of f.

♣

(6) $\int_{-1}^{2} \int_{0}^{6} x^2 \sin(x-y) dx dy = \int_{0}^{6} \int_{-1}^{2} x^2 \sin(x-y) dy dx$

Answer: True. The fact that we can interchange the order of integration is a consequence of (or, if you prefer, part of the proof of) Fubini's Theorem.

♣

(7) A rectangle of length l and width w has a fixed perimeter (i.e. perimeter = constant). Under these conditions, the rectangle of maximum area is a square.

Answer: True. This is a well-known geometric result. To prove it, we let $a(l, w) = lw$ be the area function. Now, we want to maximize a subject to the constraint that the perimeter $p(l, w) = 2l + 2w = c$ for some constant c. We can use Lagrange Multipliers:

$$\langle w, l\rangle = \nabla a = \lambda \nabla p = \lambda\langle 2, 2\rangle$$
$$p(l, w) = 2l + 2w = c$$

Hence, $w = 2\lambda$ and $l = 2\lambda$, which means that $w = l$ when a is maximized subject to the given constraint; obviously, when $l = w$, the rectangle is a square.

♣

(8) If, for some two variable function $f(x, y)$, $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$ for all real (x, y), then $f(x, y)$ is a constant.

Answer: False. Consider, for example, $f(x, y) = 2x + 2y$. Then $\frac{\partial f}{\partial x} = 2$ and $\frac{\partial f}{\partial y} = 2$, but f is certainly non-constant.

♣

Multiple Choice

(1) If R is the region inside the circle $x^2+y^2 = 4$, then $\int\int_R x\sqrt{x^2 + y^2}dA$ is equal to...

Answer: Option E, $\int_0^2 \int_0^{2\pi} r^3 \cos\theta d\theta dr$. To see this, first note that $\sqrt{x^2 + y^2} = r$ and $x = r\cos\theta$. Also, the circle $x^2 + y^2 = 4$ is given by $r^2 = 4$ or $r = 2$, with θ varying from 0 to 2π. Hence,

$$\int\int_R x\sqrt{x^2 + y^2}dA = \int_0^2 \int_0^{2\pi} r\cos\theta r d\theta r dr = \int_0^2 \int_0^{2\pi} r^3 \cos\theta d\theta dr.$$

♣

(2) Evaluate $\int_0^1 \int_0^2 (x + y) dx dy$.

Answer: Option F, 3. This is just a straight computation:

$$\begin{aligned}\int_0^1 \int_0^2 (x+y)dxdy &= \int_0^1 \left[\frac{x^2}{2} + xy\right]_0^2 dy \\ &= \int_0^1 [2+2y]\, dy \\ &= \left[2y + y^2\right]_0^1 \\ &= 3\end{aligned}$$

♣

(3) How many critical points does the function $f(x,y) = x^2+y^2+2x^2y+3$ have?

Answer: Option D, 3. We need to compute the partial derivatives:

$$\begin{aligned}f_x &= 2x + 4xy = 2x(1+2y) \\ f_y &= 2y + 2x^2\end{aligned}$$

When $f_x = 0$, either $2x = 0$ or $1+2y = 0$, so either $x = 0$ or $y = \frac{-1}{2}$. Now, when $x = 0$ and $f_y = 0$, we have that

$$0 = 2y + 2(0)^2 = 2y,$$

so $y = 0$. When $y = \frac{-1}{2}$ and $f_y = 0$, then

$$0 = 2\left(\frac{-1}{2}\right) + 2x^2 = -1 + 2x^2,$$

so $2x^2 = 1$, which in turn implies that $x = \pm\frac{1}{\sqrt{2}}$. Therefore, we have three critical points for f:

$$(0,0) \quad \left(\frac{1}{\sqrt{2}}, \frac{-1}{2}\right) \quad \left(\frac{-1}{\sqrt{2}}, \frac{-1}{2}\right).$$

♣

(4) Let S be the surface $x^2y + 4xz^3 - yz = 0$. An equation for the tangent plane to S at $(1,2,-1)$ is ...

Answer: Option A, $2y + 10z = -6$. To find the tangent plane, we first compute the gradient of $f(x,y,z) = x^2y + 4xz^3 - yz$, which will be perpendicular to the level surface $x^2y + 4xz^3 - yz = 0$:

$$\nabla f = \langle f_x, f_y, f_z\rangle = \langle 2xy + 4z^3, x^2 - z, 12xz^2 - y\rangle.$$

Therefore,

$$\nabla f(1,2,-1) = \langle 0, 2, 10\rangle.$$

Since this vector is perpendicular to the level surface at this point, it is also perpendicular to the tangent plane at the point. We know how to compute the equation of a plane through the point $(1, 2, -1)$ perpendicular to the vector $\langle 0, 2, 10\rangle$; namely, the plane is given by:

$$0(x-1) + 2(y-2) + 10(z+1) = 0,$$

or

$$2y + 10z = -6.$$

♣

(5) The point at which the function $f(x, y) = xy - x^2y - xy^2$ has a local maximum is:

Answer: Option B, $(1/3, 1/3)$. To find the local maximum, we need to compute the critical points, which means we need to determine the partial derivatives:

$$f_x = y - 2xy - y^2 = y(1 - 2x - y)$$
$$f_y = x - x^2 - 2xy = x(1 - x - 2y)$$

When $f_x = 0$, either $y = 0$ or $1 - 2x - y = 0$. On the other hand, when $f_y = 0$, either $x = 0$ or $1 - x - 2y = 0$; hence the critical points of f are $(0, 0)$ and those points satisfying the system:

$$1 - 2x - y = 0$$
$$1 - x - 2y = 0$$

From the first equation, we see that $y = 1 - 2x$; plugging that into the second equation yields:

$$0 = 1 - x - 2(1 - 2x) = 1 - x - 2 + 4x = -1 + 3x,$$

so $x = \frac{1}{3}$. In turn, this implies that $y = 1 - 2\left(\frac{1}{3}\right) = \frac{1}{3}$. Therefore, the critical points of f are

$$(0,0) \qquad \left(\frac{1}{3}, \frac{1}{3}\right).$$

Now, we need to apply the second derivative test, which means we need to compute the second partials:

$$f_{xx} = -2y$$
$$f_{xy} = 1 - 2x$$
$$f_{yy} = -2x$$

Hence,

$$f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}^2(0, 0) = 0 - 1 = -1 < 0$$

and

$$\begin{aligned} f_{xx}(1/3, 1/3)f_{yy}(1/3, 1/3) - f_{xy}^2(1/3, 1/3) &= (-2/3)(-2/3) - (1 - 2/3)^2 \\ &= 1/3 \\ &> 0. \end{aligned}$$

Since $f_{xx}(1/3, 1/3) = -2/3$, we see that $(1/3, 1/3)$ is a local maximum of f (and that the only other critical point, $(0, 0)$, is a saddle point).

♣

(6) Given $f(x, y) = x^2y^3$, $\mathbf{u} = \langle 3/5, -4/5\rangle$, the directional derivative $D_{\mathbf{u}}f$ in the direction of $\mathbf{u}$ is ...

Answer: Option D, $\frac{6xy^3-12x^2y^2}{5}$. Note, first, that $\mathbf{u}$ is a unit vector, since

$$|\mathbf{u}|^2 = (3/5)^2 + (-4/5)^2 = 9/25 + 16/25 = 1.$$

Now, remember that $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$, so we'll need to determine ∇f. To that end,

$$\nabla f = \langle 2xy^3, 3x^2y^2\rangle.$$

Hence,

$$D_{\mathbf{u}}f = \nabla f\cdot\mathbf{u} = \langle 2xy^3, 3x^2y^2\rangle\cdot\langle 3/5, -4/5\rangle = \frac{6xy^3}{5}-\frac{12x^2y^2}{5} = \frac{6xy^3 - 12x^2y^2}{5}.$$

♣

(7) An ant is placed on a flat plate whose temperature distribution is given by $T(x, y) = 3x^2 + 2xy$. If the ant's initial position is $(3, -6)$, it should walk in which direction to cool off most rapidly?

Answer: Option C, $-6\mathbf{i} - 6\mathbf{j}$. The direction in which the ant will cool off most rapidly is the direction in which T decreases fastest. Remember that the direction of fastest decrease at the point $(-3, 6)$ is given by $-\nabla T(-3, 6)$. Now,

$$\nabla T = \langle T_x, T_y\rangle = \langle 6x + 2y, 2x\rangle,$$

so

$$-\nabla T(-3, 6) = \langle 6(-3) + 2(6), 2(-3)\rangle = \langle -6, -6\rangle,$$

so the ant will cool off fastest if it walks in the direction $-6\mathbf{i} - 6\mathbf{j}$.

♣

(8) Evaluate $\int\int\int_R x^2 dV$, $R = \{(x, y, z)|0 \le x \le y, 0 \le y \le 1, 0 \le z \le 1\}$.

Answer: Option E, 1/12. Using Fubini's Theorem, we see that

$$\int\int\int_R x^2 dV = \int_0^1 \int_0^1 \int_0^y x^2 dx dy dz$$
$$= \int_0^1 \int_0^1 \left[\frac{x^3}{3}\right]_0^y dy dz$$
$$= \int_0^1 \int_0^1 \frac{y^3}{3} dy dz$$
$$= \int_0^1 \left[\frac{y^4}{12}\right]_0^1 dz$$
$$= \int_0^1 \frac{1}{12} dz$$
$$= \frac{1}{12}.$$

♣

(9) Find the maximum rate of change of $f(x, y, z) = \frac{x}{y} + \frac{y}{z}$ at the point $(4, 2, 1)$.

Answer: Option D, $\frac{\sqrt{17}}{2}$. The maximum rate of change of f at $(4, 2, 1)$ is simply given by $|\nabla f(4, 2, 1)|$. Now,

$$\nabla f = \langle f_x, f_y, f_z\rangle = \langle \frac{1}{y}, \frac{-x}{y^2} + \frac{1}{z}, \frac{-y}{z^2}\rangle.$$

Hence,

$$\nabla f(4, 2, 1) = \langle \frac{1}{2}, \frac{-4}{4} + 1, -2\rangle = \langle \frac{1}{2}, 0, -2\rangle.$$

Thus, the maximum rate of change is

$$|\nabla f(4, 2, 1)| = \sqrt{\left(\frac{1}{2}\right)^2 + (0)^2 + (-2)^2} = \sqrt{\frac{1}{4} + 4} = \sqrt{\frac{17}{4}} = \frac{\sqrt{17}}{2}.$$

♣

(10) Find the directional derivative of $f(x, y, z) = xe^{\frac{xy}{z}}$ in the direction of $\mathbf{u} = \frac{-1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$ at the point $(3, 0, 1)$.

Answer: Option A, 17/3. First, note that

$$|\mathbf{u}|^2 = \left(\frac{-1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = \frac{1}{9} + \frac{4}{9} + \frac{4}{9} = 1,$$

so $\mathbf{u}$ is a unit vector. Hence, the directional derivative in the direction of $\mathbf{u}$ is given by $\nabla f(3, 0, 1) \cdot \mathbf{u}$. Now,

$$\nabla f = \langle f_x, f_y, f_z\rangle = \langle xe^{\frac{xy}{z}}\frac{y}{z} + e^{\frac{xy}{z}}, xe^{\frac{xy}{z}}\frac{x}{z}, xe^{\frac{xy}{z}}\frac{-xy}{z^2}\rangle = \langle \frac{xy}{z}e^{\frac{xy}{z}} + e^{\frac{xy}{z}}, \frac{x^2}{z}e^{\frac{xy}{z}}, \frac{-x^2y}{z^2}e^{\frac{xy}{z}}\rangle.$$

Hence,

$$\nabla f(3, 0, 1) = \langle 1, 9, 0\rangle,$$

and so

$$\nabla f(3,0,1)\cdot \mathbf{u} = \langle 1,9,0\rangle \cdot \langle -1/3, 2/3, 2/3\rangle = \frac{-1}{3} + \frac{18}{3} = \frac{17}{3}.$$

♣

(11) The function $f(x,y) = x^2 + y^2 + xy$ has one critical point; determine the location and nature of the point.

Answer: Option B, $(0,0)$ is a local minimum. Now,

$$f_x = 2x + y$$
$$f_y = 2y + x$$

Now, if $f_x = 0$, $2x + y = 0$, so $y = -2x$. If $f_y = 0$, then

$$0 = 2y + x = 2(-2x) + x = -4x + x = -3x,$$

so $x = 0$. Hence, $y = -2(0) = 0$, so $(0,0)$ is the only critical point of f. To use the second derivative test, we need to compute the second partials:

$$f_{xx} = 2$$
$$f_{xy} = 1$$
$$f_{yy} = 2.$$

Hence, $f_{xx}f_{yy} - f_{xy}^2 = (2)(2) - 1^2 = 4 - 1 = 3 > 0$. Since $f_{xx} = 2 > 0$, this means that $(0,0)$ is a local minimum.

♣

(12) Find the minimum value of the function $f(x,y) = 2x^2 + y^2$ subject to the constraint $xy = 2$.

Answer: Option E, $4\sqrt{2}$. We let $g(x,y) = xy - 2$ and use Lagrange Multipliers:

$$\langle 4x, 2y\rangle = \nabla f = \lambda \nabla g = \langle y, x\rangle$$
$$g(x,y) = xy - 2 = 0.$$

Hence, $4x = \lambda y$ and $2y = \lambda x$. Thus, $x = \frac{\lambda y}{4}$, so

$$0 = xy - 2 = \frac{\lambda y}{4} y - 2 = \frac{\lambda}{4} y^2 - 2.$$

Note that $\lambda \neq 0$ because, if it were, we would have $x = y = 0$, which can't happen, since $g(0,0) = -2 \neq 0$. Hence $y^2 = \frac{8}{\lambda}$, so $y = \pm\frac{2\sqrt{2}}{\sqrt{\lambda}}$. In turn, this means that

$$x = \frac{\lambda y}{4} = \frac{\lambda}{4}\left(\frac{\pm 2\sqrt{2}}{\sqrt{\lambda}}\right) = \pm\frac{\sqrt{2}\sqrt{\lambda}}{2}.$$

Thus,

$$\pm\frac{2\sqrt{2}}{\sqrt{\lambda}} = y = \frac{1}{2}\lambda x = \frac{1}{2}\lambda\left(\frac{\pm\sqrt{2}\sqrt{\lambda}}{2}\right) = \pm\frac{\lambda\sqrt{\lambda}}{\sqrt{2}}.$$

Therefore, $2\sqrt{2} = \frac{\lambda^2}{2\sqrt{2}}$, so $\lambda^2 = 8$, meaning $\lambda = \pm 2\sqrt{2}$. Note that λ must be positive; if not, then x and y have opposite signs (since $4x = \lambda y$), which is impossible, since $xy = 2$. Therefore, $\lambda = 2\sqrt{2}$, so

$$x = \frac{\sqrt{\lambda}}{\sqrt{2}} = \frac{\sqrt{2\sqrt{2}}}{\sqrt{2}} = \sqrt[4]{2}$$

and

$$y = \frac{\lambda\sqrt{\lambda}}{2\sqrt{2}} = \frac{2\sqrt{2}\sqrt{2\sqrt{2}}}{2\sqrt{2}} = \sqrt{2\sqrt{2}}.$$

Therefore, f has a minimum at $(x, y) = (\sqrt[4]{2}, \sqrt{2\sqrt{2}})$; this minimum is given by

$$f\left(\sqrt[4]{2}, \sqrt{2\sqrt{2}}\right) = 2\left(\sqrt[4]{2}\right)^2 + \left(\sqrt{2\sqrt{2}}\right)^2 = 2\sqrt{2} + 2\sqrt{2} = 4\sqrt{2}.$$

♣

Free Response

(1) Convert the integral $\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} e^{-(x^2+y^2)}dydx$ to polar coordinates, then evaluate it. Sketch the region over which the integration is performed.

Answer: Since y ranges from 0 to $\sqrt{4-x^2}$, y^2 ranges from 0 to $4 - x^2$, which corresponds to the half of the disc contained in the circle $x^2 + y^2 = 4$ above the x-axis. Hence, the region of integration is:

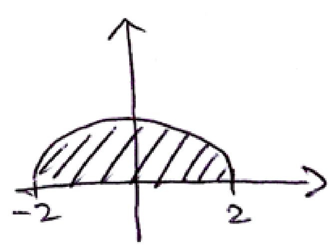

Figure 1

Now, on this region, if we view it in polar coordinates, r ranges from 0 to 2 and θ ranges from 0 to π. Also, $x^2 + y^2 = r^2$. Hence, we convert the integral to polar coordinates as:

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} e^{-(x^2+y^2)}dydx = \int_{0}^{\pi}\int_{0}^{2} e^{-r^2} r dr d\theta.$$

Now, to compute the value of this integral, let's use a u substitution, where $u = -r^2$. Then $du = -2rdr$. Hence,

$$\begin{aligned}\int_0^\pi \int_0^2 e^{-r^2} r dr d\theta &= -\frac{1}{2}\int_0^\pi \int_0^{-4} e^u du d\theta \\ &= \frac{-1}{2}\int_0^\pi [e^u]_0^{-4} d\theta \\ &= \frac{-1}{2}\int_0^\pi \left[e^{-4}-1\right] d\theta \\ &= \frac{-1}{2}\left[(e^{-4}-1)\theta\right]_0^\pi \\ &= \frac{\pi}{2}\left(1-\frac{1}{e^4}\right)\end{aligned}$$

♣

(2) Suppose that the production of a certain object depends on the availability of two raw materials, x and y according to the function $p(x, y) = x^{2/3}y^{1/3}$. The budget for the production is a fixed number of dollars, \$$c$. If each unit of material x costs \$1000 and each unit of material y costs \$1000 and the total budget is \$378,000, then the budget constraint for the production (in units of \$1000) is $x + y =$ \$378. What is the maximum production possible given the budget constraint?

Answer: Let $g(x, y) = x + y - 378$. Then the constraint is given by $g(x, y) = 0$. Now, we use Lagrange Multipliers:

$$\langle \frac{2}{3}x^{-1/3}y^{1/3}, \frac{1}{3}x^{2/3}y^{-2/3}\rangle = \nabla p = \lambda \nabla g = \lambda\langle 1, 1\rangle$$
$$g(x, y) = x + y - 378 = 0$$

Thus, we have the following system of equations:

$$\frac{2}{3}x^{-1/3}y^{1/3} = \lambda$$
$$\frac{1}{3}x^{2/3}y^{-2/3} = \lambda$$
$$x + y = 378$$

Therefore, multiplying the top two equations by 3 and setting them equal, we see that

$$2x^{-1/3}y^{1/3} = x^{2/3}y^{-2/3}.$$

Hence, multiplying both sides by $x^{1/3}$ and $y^{2/3}$, we see that

$$2y = x.$$

Therefore,

$$378 = x + y = 2y + y = 3y,$$

so $y = 126$. Therefore, $x = 2y = 252$. Therefore, maximum production occurs when $(x, y) = (252, 126)$ and maximum production is

$$p(252, 126) = 252^{2/3}126^{1/3} = 126\sqrt[3]{4}.$$

♣

(3) Find any global maxima or minima of $h(x, y) = 1 + x^2 + y^2$ on the disk $x^2 + y^2 \leq 1$.

Answer: Remember that absolute maxima and minima of h on a region will occur at critical points or on the boundary. Hence, we first need to compute critical points, which will occur when

$$0 = h_x = 2x$$
$$0 = h_y = 2y,$$

so the only critical point of h is at $(0, 0)$. $h(0, 0) = 1$. Now, we consider boundary points. The boundary of the disk is the circle given by $x^2 + y^2 = 1$. Therefore, if (x, y) is on the boundary,

$$h(x, y) = 1 + x^2 + y^2 = 1 + 1 = 2.$$

Therefore, $(0, 0)$ is the absolute minimum of h on this region and every point on the boundary is an absolute maximum of h on the region.

♣

(4) Consider the integral $\int_0^3 \int_0^{\frac{1}{2}(3-z)} \int_0^{4-x^2} dydxdz$.

(a) Sketch the solid whose volume is given by the integral.

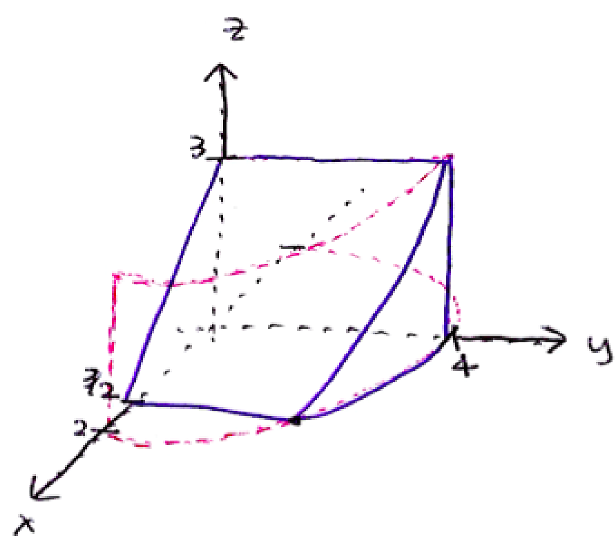

FIGURE 2

The dotted pink curves denote the paraboloid given by $4 - x^2$.

(b) Evaluate the integral.

Answer:

$$\int_0^3 \int_0^{\frac{1}{2}(3-z)} \int_0^{4-x^2} dydxdz = \int_0^3 \int_0^{\frac{1}{2}(3-z)} [y]_0^{4-x^2}\, dxdz$$
$$= \int_0^3 \int_0^{\frac{1}{2}(3-z)} (4-x^2) dxdz$$
$$= \int_0^3 \left[4x - \frac{x^3}{3}\right]_0^{\frac{1}{2}(3-z)} dz$$
$$= \int_0^3 \left[2(3-z) - \frac{1}{24}(3-z)^3\right] dz.$$

Make the substitution $u = 3 - z$; then $du = -dz$. Hence,

$$\int_0^3 \left[2(3-z) - \frac{1}{24}(3-z)^3\right] dz = -\int_3^0 \left[2u - \frac{1}{24}u^3\right] du$$
$$= \int_0^3 \left[2u - \frac{1}{24}u^3\right] du$$
$$= \left[u^2 - \frac{u^4}{96}\right]_0^3$$
$$= 9 - \frac{81}{96}$$
$$= 9 - \frac{27}{32}$$
$$= \frac{261}{32}.$$

♣

DRL 3E3A, University of Pennsylvania
E-mail address: shonkwil@math.upenn.edu